GROUP ACTIONS AND A MULTI-PARAMETER FALCONER DISTANCE PROBLEM

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Abstract. In this paper we study the following multi-parameter variant of the celebrated Falconer distance problem ([6]). Given $\mathbf{d} = (d_1, d_2, \ldots, d_\ell) \in \mathbb{N}^\ell$ with $d_1 + d_2 + \cdots + d_\ell = d$ and $E \subseteq \mathbb{R}^d$, we define

$$\Delta_{\mathbf{d}}(E) = \left\{ \left(|x^{(1)} - y^{(1)}|, \ldots, |x^{(\ell)} - y^{(\ell)}| \right) : x, y \in E \right\} \subseteq \mathbb{R}^\ell,$$

where for $x \in \mathbb{R}^d$ we write $x = \left(x^{(1)}, \ldots, x^{(\ell)}\right)$ with $x^{(i)} \in \mathbb{R}^{d_i}$.

We ask how large the Hausdorff dimension of E needs to be to ensure that the ℓ-dimensional Lebesgue measure of $\Delta_{\mathbf{d}}(E)$ is positive. We prove that if $2 \le d_i$ for $1 \le i \le \ell$, then the conclusion holds provided

$$\dim(E) > d - \frac{\min d_i}{2} + \frac{1}{3}.$$

We also note that, by previous constructions, the conclusion does not in general hold if

$$\dim(E) < d - \frac{\min d_i}{2}.$$

A group action derivation of a suitable Mattila integral plays an important role in the argument.

1. Introduction

Given a set $E \subseteq \mathbb{R}^d$, the distance set of E is

$$\Delta(E) = \{|x - y| : x, y \in E\} \subseteq \mathbb{R}.$$

Falconer [6] studied how large the Hausdorff dimension of E must be to guarantee that the Lebesgue measure of $\Delta(E)$ is positive. Falconer's conjecture is

Conjecture 1.1. *Let E be a compact subset of $\mathbb{R}^d$, $d \ge 2$. If $\dim(E) > d/2$, then $|\Delta(E)| > 0$.*

Here $|\cdot|$ is the Lebesgue measure and $\dim(\cdot)$ is the Hausdorff dimension. In [6], Falconer showed that $d/2$ in the conjecture is best possible by constructing, for each $0 < s < d/2$, a compact set $E_s \subseteq \mathbb{R}^d$ such that $\dim(E_s) = s$ and $\dim(\Delta(E_s)) \le 2s/d$. Falconer's conjecture is open for all dimensions $d \ge 2$. Partial results have been obtained by Falconer [6], Mattila [12], Bourgain [2], and others. The best currently known result, due to Wolff [14] ($d = 2$) and Erdoğan [5] ($d \ge 3$), is

Theorem 1.2. *Let E be a compact subset of $\mathbb{R}^d$, $d \ge 2$. If $\dim(E) > d/2 + 1/3$, then $|\Delta(E)| > 0$.*

We will study a multi-parameter variant of Falconer's distance problem. Given $\mathbf{d} = (d_1, \ldots, d_\ell) \in \mathbb{N}^\ell$, we let $d = d_1 + \cdots + d_\ell$. For $x \in \mathbb{R}^d$, we write

$$x = \left(x^{(1)}, \ldots, x^{(\ell)}\right)$$

where $x^{(i)} \in \mathbb{R}^{d_i}$. Given a set $E \subseteq \mathbb{R}^d$, we define the multi-parameter distance set of E to be

$$\Delta_{\mathbf{d}}(E) = \left\{ \left(|x^{(1)} - y^{(1)}|, \ldots, |x^{(\ell)} - y^{(\ell)}| \right) : x, y \in E \right\} \subseteq \mathbb{R}^\ell.$$

Further, we let

$$\mathcal{F}(\mathbf{d}) = \sup \left\{ \dim(E) : E \subseteq \mathbb{R}^d,\ |\Delta_{\mathbf{d}}(E)| = 0 \right\}.$$

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By considering (a sequence of near) maximal dimensional sets with zero-measure distance sets in one hyperplane, crossed with full boxes in the other hyperplanes, we immediately have the relation

$$\mathcal{F}(\mathbf{d}) \geq d - d_i + \mathcal{F}(d_i)$$

for all $1 \leq i \leq \ell$. Moreover, by the construction of Falconer [6] mentioned above, we have $\mathcal{F}(d_i) \geq d_i/2$ for all $1 \leq i \leq \ell$, and so

$$\mathcal{F}(\mathbf{d}) \geq d - \frac{\min d_i}{2}.$$

Our main result is

Theorem 1.3. *Let $\mathbf{d} = (d_1, \ldots, d_\ell) \in \mathbb{N}^\ell$ with $2 \leq d_i$ for $1 \leq i \leq \ell$ and $d = d_1 + \cdots + d_\ell$. If E is a compact subset of $\mathbb{R}^d$ with*

$$\dim(E) > d - \frac{\min d_i}{2} + \frac{1}{3}, \tag{1.1}$$

then $|\Delta_{\mathbf{d}}(E)| > 0$.

In other words, Theorem 1.3 is precisely the statement that

$$\mathcal{F}(\mathbf{d}) \leq d - \frac{\min d_i}{2} + \frac{1}{3}.$$

Note that Theorem 1.3 recovers Theorem 1.2 by taking $\ell = 1$. Note also that a similar problem has been studied in vector spaces over finite fields by Birklbauer and Iosevich [1].

The standard approach in studying Falconer's distance conjecture and related problems is to reduce the problem to the estimation of a so-called Mattila integral. This reduction is typically carried out via a stationary phase argument (see, for example, [2], [5], [12], [13], [14], and references therein). Our approach is notable in that we instead carry out this reduction via a group action method developed by Greenleaf, Iosevich, Liu, and Palsson [9] in the study of the distribution of simplexes in compact sets of a given Hausdorff dimension. Liu [11] has given a general formulation of the method and used it to study continuous sum-product problems. The method has its roots in the method developed by Elekes and Sharir in [4], which was ultimately used by Guth and Katz [10] to prove the Erdős distance conjecture in the plane. We estimate the Mattila integral by an iterative argument that combines a "Fourier slicing" technique and a strong bound on the L^2 spherical average of the Fourier transform of a measure on $\mathbb{R}^n$ due to Wolff [14] ($n = 2$) and Erdoğan [5] ($n \geq 3$).

2. Proof of Theorem 1.3

For the entirety of the proof, we fix $\mathbf{d} = (d_1, \ldots, d_\ell) \in \mathbb{N}^\ell$ with $2 \leq d_i$ for $1 \leq i \leq \ell$ and $d = d_1 + \cdots + d_\ell$. We also fix a compact set $E \subseteq \mathbb{R}^d$.

The notation $A \lesssim B$ means there is a constant $C > 0$ such that $A \leq CB$; the constant may depend on $(d_1, \ldots, d_\ell)$ and E, but not on any other parameters. Additionally, $A \gtrsim B$ means $B \lesssim A$, and $A \approx B$ means both $A \lesssim B$ and $B \lesssim A$. For $n \in \mathbb{N}$, we let $\mathbb{O}(n)$ denote the orthogonal group on $\mathbb{R}^n$, and we note that $\mathbb{O}(n)$ is a compact group with the operator norm topology.

For each finite non-negative Borel measure μ supported on E, we define a measure ν on $\mathbb{R}^\ell$ by

$$\int_{\mathbb{R}^\ell} f(t) d\nu(t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(|x^{(1)} - y^{(1)}|, \ldots, |x^{(\ell)} - y^{(\ell)}|) d\mu(x) d\mu(y),$$

and, further, for each $\mathbf{g} = (g^{(1)}, \ldots, g^{(\ell)}) \in \prod_{i=1}^{\ell} \mathbb{O}(d_i)$, we define a measure $\nu_{\mathbf{g}}$ on $\mathbb{R}^d$ by

$$\int_{\mathbb{R}^d} f(z) d\nu_{\mathbf{g}}(z) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x^{(1)} - g^{(1)} y^{(1)}, \ldots, x^{(\ell)} - g^{(\ell)} y^{(\ell)}) d\mu(x) d\mu(y).$$

We emphasize that ν and $\nu_{\mathbf{g}}$ both depend on μ and that $\operatorname{supp}(\nu) \subseteq \Delta_{\mathbf{d}}(E)$.

Our goal is to show that, whenever (1.1) holds, there is a choice of μ for which the Fourier transform $\widehat{\nu}$ is in L^2. This will imply ν has an L^2 density with respect to Lebesgue measure on $\mathbb{R}^\ell$, and hence $|\Delta_{\mathbf{d}}(E)| > 0$.

Our argument has two parts. In the first part, we exploit the action of the orthogonal group to show that, for any measure μ as above,

$$\int_{\mathbb{R}^\ell} |\widehat{\nu}(\eta)|^2 d\eta \lesssim \int_{\mathbb{R}^d} \int_{\prod_{i=1}^\ell \mathbb{O}(d_i)} |\widehat{\nu_{\mathbf{g}}}(\xi)|^2 d\mathbf{g} d\xi \approx \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 \int_{\prod_{i=1}^\ell S^{d_i-1}} |\widehat{\mu}(|\xi^{(1)}|\theta^{(1)}, \dots, |\xi^{(\ell)}|\theta^{(\ell)})|^2 d\boldsymbol{\theta} d\xi.$$

This is split into Lemma 2.1 and Lemma 2.2. Here $d\mathbf{g} = dg^{(1)} \cdots dg^{(\ell)}$ is the product of the normalized Haar measures on $\mathbb{O}(d_i)$, $i = 1, \dots, \ell$, and $d\boldsymbol{\theta} = d\theta^{(1)} \cdots d\theta^{(\ell)}$ is the product of the uniform probability measures on the spheres S^{d_i-1}, $i = 1, \dots, \ell$.

In the second part of the argument, we use a slicing technique and a bound due to Wolff [14] ($n = 2$) and Erdoğan [5] ($n \geq 3$) on the L^2 spherical average of the Fourier transform of a measure on $\mathbb{R}^n$ to show that the multi-parameter Mattila integral

$$\int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 \int_{\prod_{i=1}^\ell S^{d_i-1}} |\widehat{\mu}(|\xi^{(1)}|\theta^{(1)}, \dots, |\xi^{(\ell)}|\theta^{(\ell)})|^2 d\boldsymbol{\theta} d\xi$$

is finite for some Frostman measure μ on E whose existence is implied by the dimension hypothesis (1.1). This is Lemma 2.3.

2.1. **Exploiting the Action of the Orthogonal Group.**

Lemma 2.1. *For any finite non-negative Borel measure μ supported on E,*

$$\int_{\mathbb{R}^\ell} |\widehat{\nu}(\eta)|^2 d\eta \lesssim \int_{\mathbb{R}^d} \int_{\prod_{i=1}^\ell \mathbb{O}(d_i)} |\widehat{\nu_{\mathbf{g}}}(\xi)|^2 d\mathbf{g} d\xi.$$

Proof. We begin by fixing approximate identities on $\mathbb{R}^\ell$ and $\mathbb{R}^d$ as follows. We choose $\phi \in C_c^\infty(\mathbb{R}^\ell)$ with $\phi \geq 0$, $\text{supp}(\phi) \subseteq [-1, 1]^\ell$, and $\int \phi(x) dx = 1$, and the associated approximate identity is $\phi_\epsilon(x) = \epsilon^{-\ell} \phi(\epsilon^{-1} x)$ for $\epsilon > 0$. Similarly, we choose $\psi \in C_c^\infty(\mathbb{R}^d)$ with $\psi \geq 0$, $\text{supp}(\phi) \subseteq [-1, 1]^d$, $\int \psi(x) dx = 1$, and $\psi \geq \frac{1}{2}$ on $[-\frac{1}{2}, \frac{1}{2}]^d$, and the associated approximate identity is $\psi_\epsilon(x) = \epsilon^{-d} \psi(\epsilon^{-1} x)$ for $\epsilon > 0$.

Since $\widehat{\phi_\epsilon * \nu} \to \widehat{\nu}$ and $\widehat{\psi_\epsilon * \nu_{\mathbf{g}}} \to \widehat{\nu_{\mathbf{g}}}$ uniformly as $\epsilon \to 0$, Plancherel's theorem tells us that Lemma 2.1 will be proved upon establishing that, for all $\epsilon > 0$,

$$\int_{\mathbb{R}^\ell} (\phi_\epsilon * \nu)^2(t) dt \lesssim \int_{\mathbb{R}^d} \int_{\prod_{i=1}^\ell \mathbb{O}(d_i)} (\psi_{c\epsilon} * \nu_{\mathbf{g}})^2(z) d\mathbf{g} dz, \tag{2.1}$$

where $c > 0$ is a constant depending only on the diameter of E.

For $\epsilon > 0$ and $\mathbf{g} \in \prod_{i=1}^\ell \mathbb{O}(d_i)$, we define the sets

$$D(\epsilon) = \left\{ (u, v, x, y) \in E^4 : \left| |x^{(i)} - y^{(i)}| - |u^{(i)} - v^{(i)}| \right| \leq \epsilon \quad \forall\, 1 \leq i \leq \ell \right\},$$
$$G(\epsilon, \mathbf{g}) = \left\{ (u, v, x, y) \in E^4 : |x^{(i)} - y^{(i)} - g^{(i)}(u^{(i)} - v^{(i)})| \leq \epsilon \quad \forall\, 1 \leq i \leq \ell \right\}.$$

We will establish (2.1) by proving the following three inequalities:

$$\int_{\mathbb{R}^\ell} (\phi_\epsilon * \nu)^2(t) dt \lesssim \epsilon^{-\ell} \mu^4(D(2\epsilon)), \tag{2.2}$$

$$\epsilon^{-\ell} \mu^4(D(\epsilon)) \lesssim \epsilon^{-d} \int_{\prod_{i=1}^\ell \mathbb{O}(d_i)} \mu^4(G(c\epsilon, \mathbf{g})) d\mathbf{g}, \tag{2.3}$$

$$\epsilon^{-d} \mu^4(G(\epsilon/4, \mathbf{g})) \lesssim \int_{\mathbb{R}^d} (\psi_\epsilon * \nu_{\mathbf{g}})^2(z) dz, \tag{2.4}$$

where μ^4 denotes the product measure $\mu \times \mu \times \mu \times \mu$, and $c = 2 \max\{2\text{diam}(E), 1\}$ in (2.3).

We start by proving (2.2).

For $t \in \mathbb{R}^\ell$, we have

$$\begin{aligned}
\phi_\epsilon * \nu(t) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi_\epsilon\left(t_1 - |x^{(1)} - y^{(1)}|, \ldots, t_\ell - |x^{(\ell)} - y^{(\ell)}|\right) d\mu(x)d\mu(y) \\
&\lesssim \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \epsilon^{-\ell} \prod_{i=1}^{\ell} \chi\left\{\left|t_i - |x^{(i)} - y^{(i)}|\right| \leq \epsilon\right\} d\mu(x)d\mu(y),
\end{aligned}$$

where χA denotes the indicator function of a set A. Therefore, by the triangle inequality,

$$\begin{aligned}
&\int_{\mathbb{R}^\ell} (\phi_\epsilon * \nu)^2(t)dt \\
&\lesssim \epsilon^{-2\ell} \int \prod_{i=1}^{\ell} \chi\left\{\left|t_i - |x^{(i)} - y^{(i)}|\right| \leq \epsilon\right\} \chi\left\{\left|t_i - |u^{(i)} - v^{(i)}|\right| \leq \epsilon\right\} d\mu^4(u,v,x,y)dt \\
&\leq \epsilon^{-2\ell} \int \prod_{i=1}^{\ell} \chi\left\{\left|t_i - |x^{(i)} - y^{(i)}|\right| \leq \epsilon\right\} \chi\left\{\left||x^{(i)} - y^{(i)}| - |u^{(i)} - v^{(i)}|\right| \leq 2\epsilon\right\} d\mu^4(u,v,x,y)dt
\end{aligned}$$

For fixed $x^{(i)}, y^{(i)} \in \mathbb{R}^{d_i}$, the set of $t_i \in \mathbb{R}$ with $\left|t_i - |x^{(i)} - y^{(i)}|\right| \leq \epsilon$ has Lebesgue measure $\approx \epsilon$. Thus integrating out dt in the last integral yields (2.2).

Now we prove (2.4).

Our choice of ψ guarantees that $\psi_\epsilon \geq \frac{1}{2}\epsilon^{-d}$ on $[-\frac{1}{2}\epsilon, \frac{1}{2}\epsilon]^d$. Thus, for all $z \in \mathbb{R}^d$,

$$\begin{aligned}
\psi_\epsilon * \nu_{\mathbf{g}}(z) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi_\epsilon(z^{(1)} - (x^{(1)} - g^{(1)}y^{(1)}), \ldots, z^{(\ell)} - (x^{(\ell)} - g^{(\ell)}y^{(\ell)}))d\mu(x)d\mu(y) \\
&\gtrsim \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \epsilon^{-d} \prod_{i=1}^{\ell} \chi\left\{|z^{(i)} - (x^{(i)} - g^{(i)}y^{(i)})| \leq \frac{\epsilon}{2}\right\} d\mu(x)d\mu(y).
\end{aligned}$$

Therefore, by the triangle inequality,

$$\begin{aligned}
&\int_{\mathbb{R}^d} (\psi_\epsilon * \nu_{\mathbf{g}})^2(z)dz \\
&\gtrsim \epsilon^{-2d} \int \prod_{i=1}^{\ell} \chi\left\{|z^{(i)} - (x^{(i)} - g^{(i)}u^{(i)})| \leq \frac{\epsilon}{2}\right\} \chi\left\{|z^{(i)} - (y^{(i)} - g^{(i)}v^{(i)})| \leq \frac{\epsilon}{2}\right\} d\mu^4(u,v,x,y)dz \\
&\geq \epsilon^{-2d} \int \prod_{i=1}^{\ell} \chi\left\{|z^{(i)} - (x^{(i)} - g^{(i)}u^{(i)})| \leq \frac{\epsilon}{4}\right\} \chi\left\{|(x^{(i)} - g^{(i)}u^{(i)}) - (y^{(i)} - g^{(i)}v^{(i)})| \leq \frac{\epsilon}{4}\right\} d\mu^4(u,v,x,y)dz \\
&= \epsilon^{-2d} \int \prod_{i=1}^{\ell} \chi\left\{|z^{(i)} - (x^{(i)} - g^{(i)}u^{(i)})| \leq \frac{\epsilon}{4}\right\} \chi\left\{|x^{(i)} - y^{(i)} - g^{(i)}(u^{(i)} - v^{(i)})| \leq \frac{\epsilon}{4}\right\} d\mu^4(u,v,x,y)dz.
\end{aligned}$$

For fixed $x^{(i)}, u^{(i)} \in \mathbb{R}^{d_i}$ and $g^{(i)} \in \mathbb{O}(d_i)$, the set of $z^{(i)} \in \mathbb{R}^{d_i}$ with $|z^{(i)} - (x^{(i)} - g^{(i)}u^{(i)})| \leq \epsilon/4$ has Lebesgue measure $\approx \epsilon^{d_i}$. Thus integrating out dz in the last integral yields (2.4).

Finally we prove (2.3).

Consider a fixed $1 \leq i \leq \ell$. For the action of $\mathbb{O}(d_i)$ on $\mathbb{R}^{d_i}$, the orbit of e_{d_i} is $\text{Orb}(e_{d_i}) = \{ge_{d_i} : g \in \mathbb{O}(d_i)\} = S^{d_i-1}$. We view the sphere S^{d_i-1} as a metric space with the Euclidean metric from $\mathbb{R}^{d_i}$. We fix a cover of S^{d_i-1} by balls of radius ϵ such that the number of balls in the cover is $N(\epsilon, i) \approx \epsilon^{-(d_i-1)}$ and such that the cover has bounded overlap (that is, each set in the cover intersects no more than C other sets in the cover, where C is a constant independent of ϵ). We let $T^{(i)}_{m_i}$ for $m_i = 1, \ldots, N(\epsilon, i)$ denote the preimages of the balls with respect to the orbit map $g \mapsto ge_{d_i}$ from $\mathbb{O}(d)$ to S^{d_i-1}. Of course, the cover $\left\{T^{(i)}_{m_i} : 1 \leq m_i \leq N(\epsilon, i)\right\}$ of $\mathbb{O}(d_i)$ also has bounded overlap. Moreover, since the image of the Haar measure on $\mathbb{O}(d_i)$ with respect to the orbit map is exactly the uniform probability measure on S^{d_i-1}, each $T^{(i)}_{m_i}$ has measure $\approx \epsilon^{d_i-1}$.

For each non-zero $w \in \mathbb{R}^{d_i}$, we define the conjugation (change of basis) map $\zeta_w : \mathbb{O}(d_i) \to \mathbb{O}(d_i)$ by $\zeta_w(g) = pgp^{-1}$, where p is a fixed but arbitrary transformation in $\mathbb{O}(d_i)$ such that $pe_{d_i} = w/|w|$. For each $\epsilon > 0$ and $\mathbf{g} \in \prod_{i=1}^{\ell} \mathbb{O}(d_i)$, we define

$$M(\epsilon) = \left\{(m_1, \ldots, m_\ell) \in \mathbb{N}^\ell : 1 \leq m_i \leq N(\epsilon, i) \quad \forall 1 \leq i \leq \ell\right\},$$
$$G'(\epsilon, \mathbf{g}) = \left\{(u, v, x, y) \in E^4 : |(x^{(i)} - y^{(i)}) - \zeta_{u^{(i)}-v^{(i)}}(g^{(i)})(u^{(i)} - v^{(i)})| \leq \epsilon \quad \forall 1 \leq i \leq \ell\right\}.$$

Claim. For any collection of transformations $g^{(i)}_{m_i} \in T^{(i)}_{m_i}$, $1 \leq i \leq \ell$, $1 \leq m_i \leq N(\epsilon, i)$, we have

$$D(\epsilon) \subseteq \bigcup_{m \in M(\epsilon)} G'(c\epsilon, \mathbf{g}_m),$$

where $c = 2\max\{2\text{diam}(E), 1\}$ and $\mathbf{g}_m = (g^{(1)}_{m_1}, \ldots, g^{(\ell)}_{m_\ell})$.

Proof of Claim. Let $u, v, x, y \in E$. It suffices to consider a fixed $1 \leq i \leq \ell$. Let $w = u^{(i)} - v^{(i)}$ and $z = x^{(i)} - y^{(i)}$. Assume $||z| - |w|| < \epsilon$. If $w = 0$ or $z = 0$, then $|z - gw| = ||z| - |w|| < \epsilon$ for all $g \in \mathbb{O}(d_i)$, and we are done. Assume w and z are non-zero. Choose $g \in \mathbb{O}(d_i)$ such that $g(w/|w|) = z/|z|$, and hence $|z - gw| = ||z| - |w|| < \epsilon$. Define $g_0 = \zeta_w^{-1}(g)$. We know $g_0 \in T^{(i)}_{m_i}$ for some $1 \leq m_i \leq N(\epsilon, i)$. Since $g^{(i)}_{m_i} \in T^{(i)}_{m_i}$ also, we have $|g_0 e_{d_i} - g^{(i)}_{m_i} e_{d_i}| < 2\epsilon$. By the definition of ζ_w, the previous inequality is equivalent to $|gw - \zeta_w(g^{(i)}_{m_i})w| < 2|w|\epsilon$. Therefore, by the triangle inequality, $|z - \zeta_w(g^{(i)}_{m_i})w| \leq \epsilon + 2|w|\epsilon \leq 2\max\{2|w|, 1\}\epsilon$. To conclude, we note that $|w| = |u^{(i)} - v^{(i)}| \leq |u - v| \leq \text{diam}(E)$. □

For each $m \in M(\epsilon)$, we choose $\mathbf{g}_m = (g^{(1)}_{m_1}, \ldots, g^{(\ell)}_{m_\ell}) \in \prod_{i=1}^{\ell} T^{(i)}_{m_i}$ such that

$$\mu^4(G'(c\epsilon, \mathbf{g_m})) \lesssim \epsilon^{-(d_1-1)} \cdots \epsilon^{-(d_\ell-1)} \int_{\prod_{i=1}^{\ell} T^{\epsilon,i}_{m_i}} \mu^4(G'(c\epsilon, \mathbf{g}))\, d\mathbf{g}.$$

Such a choice is possible because the average of a set must be larger than at least one element of the set. Then, using that $\epsilon^{-\ell}\epsilon^{-(d_1-1)} \cdots \epsilon^{-(d_\ell-1)} = \epsilon^{-d}$, the claim implies

$$\epsilon^{-\ell}\mu^4(D(\epsilon)) \leq \epsilon^{-\ell} \sum_{m \in M(\epsilon)} \mu^4(G'(c\epsilon, \mathbf{g_m})) \lesssim \epsilon^{-d} \sum_{\mathbf{m} \in M(\epsilon)} \int_{\prod_{i=1}^{\ell} T^{\epsilon,i}_{m_i}} \mu^4(G'(c\epsilon, \mathbf{g}))\, d\mathbf{g}.$$

Expanding things out, the integral on the right equals

$$\int_{\prod_{i=1}^{\ell} T^{\epsilon,i}_{m_i}} \int_{E^4} \chi\left\{|(x^{(i)} - y^{(i)}) - \zeta_{u^{(i)}-v^{(i)}}(g^{(i)})(u^{(i)} - v^{(i)})| \leq c\epsilon \quad \forall 1 \leq i \leq \ell\right\} d\mu^4(u, v, x, y) d\mathbf{g}$$
$$= \int_{E^4} \int_{\prod_{i=1}^{\ell} \zeta_{u^{(i)}-v^{(i)}}(T^{\epsilon,i}_{m_i})} \chi\left\{|(x^{(i)} - y^{(i)}) - g^{(i)}(u^{(i)} - v^{(i)})| \leq c\epsilon \quad \forall 1 \leq i \leq \ell\right\} d\mathbf{g} d\mu^4(u, v, x, y).$$

Thus, noting that $\left\{\zeta_w(T^{(i)}_{m_i}) : 1 \leq m_i \leq N(\epsilon, i)\right\}$ is a bounded overlap cover of $\mathbb{O}(d_i)$ for each non-zero $w \in \mathbb{R}^{d_i}$, we obtain

$$\epsilon^{-\ell}\mu^4(D(\epsilon)) \lesssim \epsilon^{-d} \int_{E^4} \int_{\prod_{i=1}^{\ell} \mathbb{O}(d_i)} \chi\left\{|(x^{(i)} - y^{(i)}) - g^{(i)}(u^{(i)} - v^{(i)})| \leq c\epsilon \quad \forall 1 \leq i \leq \ell\right\} d\mathbf{g} d\mu^4(u, v, x, y)$$
$$= \epsilon^{-d} \int_{\prod_{i=1}^{\ell} \mathbb{O}(d_i)} \mu^4(G(c\epsilon, \mathbf{g})) d\mathbf{g}.$$

□

Lemma 2.2. *For any finite non-negative Borel measure μ supported on E,*

$$\int_{\prod_{i=1}^{\ell} \mathbb{O}(d_i)} \int_{\mathbb{R}^d} |\widehat{\nu_{\mathbf{g}}}(\xi)|^2 d\xi d\mathbf{g} \approx \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 \int_{\prod_{i=1}^{\ell} S^{d_i-1}} |\widehat{\mu}(|\xi^{(1)}|\theta^{(1)}, \ldots, |\xi^{(\ell)}|\theta^{(\ell)})|^2 d\boldsymbol{\theta} d\xi.$$

Proof. By the definition of $\nu_{\mathbf{g}}$, we have

$$\widehat{\nu_{\mathbf{g}}}(\xi) = \widehat{\mu}(\xi)\widehat{\mu}(-(g^{(1)})^T\xi^{(1)}, \dots, -(g^{(\ell)})^T\xi^{(\ell)}),$$

where T indicates transpose. Therefore

$$\int_{\mathbb{R}^d}\int_{\prod_{i=1}^{\ell}\mathbb{O}(d_i)} |\widehat{\nu_{\mathbf{g}}}(\xi)|^2 d\mathbf{g}d\xi = \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 \int_{\prod_{i=1}^{\ell}\mathbb{O}(d_i)} |\widehat{\mu}(g^{(1)}\xi^{(1)}, \dots, g^{(\ell)}\xi^{(\ell)})|^2 d\mathbf{g}d\xi.$$

We now consider the inner integral on the right for fixed non-zero $\xi \in \mathbb{R}^d$. By a change of variable and the translation invariance of the Haar measures,

$$\int_{\prod_{i=1}^{\ell}\mathbb{O}(d_i)} |\widehat{\mu}(g^{(1)}\xi^{(1)}, \dots, g^{(\ell)}\xi^{(\ell)})|^2 d\mathbf{g} = \int_{\prod_{i=1}^{\ell}\mathbb{O}(d_i)} |\widehat{\mu}(g^{(1)}e_{d_1}|\xi^{(1)}|, \dots, g^{(\ell)}e_{d_\ell}|\xi^{(\ell)})|^2 d\mathbf{g},$$

where $e_{d_i} = (0, \dots, 0, 1) \in \mathbb{R}^{d_i}$. The stabilizer subgroup of $\mathbb{O}(d_i)$ for e_{d_i} is $\mathrm{Stab}(e_{d_i}) = \{g \in \mathbb{O}(d_i) : ge_{d_i} = e_{d_i}\}$. As $\mathbb{O}(d_i)$ is compact and $\mathrm{Stab}(e_{d_i})$ is closed, $\mathrm{Stab}(e_{d_i})$ is compact. We equip $\mathrm{Stab}(e_{d_i})$ with its normalized Haar measure. The quotient space $\mathbb{O}(d_i)/\mathrm{Stab}(e_{d_i})$ is homeomorphic to the sphere S^{d_i-1}. The measure on $\mathbb{O}(d_i)/\mathrm{Stab}(e_{d_i})$ is the image of the uniform probability measure on S^{d_i-1}; it is a left-invariant Radon measure. Putting all this together, by the quotient integral formula (see, for example, [3], [8]), the last integral above equals a constant multiple of

$$\begin{aligned}
&\int_{\prod_{i=1}^{\ell}\mathbb{O}(d_i)/\mathrm{Stab}(e_{d_i})}\int_{\prod_{i=1}^{\ell}\mathrm{Stab}(e_{d_i})} |\widehat{\mu}(g^{(1)}h^{(1)}e_{d_1}|\xi^{(1)}|, \dots, g^{(\ell)}h^{(\ell)}e_{d_\ell}|\xi^{(\ell)}|)|^2 d\mathbf{h}d\mathbf{g} \\
&= \int_{\prod_{i=1}^{\ell}\mathbb{O}(d_i)/\mathrm{Stab}(e_{d_i})}\int_{\prod_{i=1}^{\ell}\mathrm{Stab}(e_{d_i})} |\widehat{\mu}(g^{(1)}e_{d_1}|\xi^{(1)}|, \dots, g^{(\ell)}e_{d_\ell}|\xi^{(\ell)}|)|^2 d\mathbf{h}d\mathbf{g} \\
&= \int_{\prod_{i=1}^{\ell}\mathbb{O}(d_i)/\mathrm{Stab}(e_{d_i})} |\widehat{\mu}(g^{(1)}e_{d_1}|\xi^{(1)}|, \dots, g^{(\ell)}e_{d_\ell}|\xi^{(\ell)}|)|^2 d\mathbf{g} \\
&= \int_{\prod_{i=1}^{\ell}S^{d_i-1}} |\widehat{\mu}(|\xi^{(1)}|\theta^{(1)}, \dots, |\xi^{(\ell)}|\theta^{(\ell)})|^2 d\boldsymbol{\theta}.
\end{aligned}$$

□

2.2. Estimating the Multi-Parameter Mattila Integral.

Lemma 2.3. *If*

$$\dim(E) > d - \frac{\min d_i}{2} + \frac{1}{3},$$

then there exists a finite non-negative Borel measure μ supported on E satisfying

$$\int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 \int_{\prod_{i=1}^{\ell}S^{d_i-1}} |\widehat{\mu}(|\xi^{(1)}|\theta^{(1)}, \dots, |\xi^{(\ell)}|\theta^{(\ell)})|^2 d\boldsymbol{\theta}d\xi < \infty. \tag{2.5}$$

For the proof of Lemma 2.3, we need two lemmas. The first is an estimate for the L^2 spherical average of the Fourier transform due to Wolff [14] ($n = 2$) and Erdoğan [5] ($n \geq 3$).

Lemma 2.4. *Let λ be a finite compactly supported Borel measure on $\mathbb{R}^n$. If $t, \epsilon > 0$ and*

$$\frac{n}{2} \leq \alpha \leq \frac{n+2}{2},$$

then

$$\int_{S^{n-1}} |\widehat{\lambda}(t\theta)|^2 d\theta \leq C_\epsilon t^{-\frac{n+2\alpha-2}{4}+\epsilon} I_\alpha(\lambda),$$

where

$$I_\alpha(\lambda) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^{-\alpha} d\lambda(x)d\lambda(y) = C_{n,\alpha}\int_{\mathbb{R}^n} |\widehat{\lambda}(\xi)|^2|\xi|^{-n+\alpha}d\xi. \tag{2.6}$$

The second lemma that we need examines the behavior of Frostman-type measures on Cartesian products of differently-sized balls from coordinate hyperplanes. Here we let $B_\delta^d(x)$ denote the ball in $\mathbb{R}^d$ of radius δ centered at x, while we let $R_\delta^d(x)$ denote the box $x+[-\delta,\delta]^d \subseteq \mathbb{R}^d$.

Lemma 2.5. *Suppose $0 < s \le d$ and μ is a finite Borel measure on $\mathbb{R}^d$ satisfying*

$$\mu\left(B_\delta^d(x)\right) \lesssim \delta^s$$

for all $x \in \mathbb{R}^d$ and $\delta > 0$. If $x = (x^{(1)}, \ldots, x^{(\ell)}) \in \mathbb{R}^d$ and $\delta_1, \ldots, \delta_\ell > 0$ with $\delta_j \le \delta_i$ for all $1 \le i \le \ell$, then

$$\mu\left(B_{\delta_1}^{d_1}(x^{(1)}) \times \cdots \times B_{\delta_\ell}^{d_\ell}(x^{(\ell)})\right) \lesssim \delta_j^{s-(d-d_j)} \prod_{i \ne j} \delta_i^{d_i}.$$

In particular, if $x = (x^{(1)}, \ldots, x^{(\ell)}) \in \mathbb{R}^d$ and $0 < \delta_1, \ldots, \delta_\ell \le 1$, then

$$\mu\left(B_{\delta_1}^{d_1}(x^{(1)}) \times \cdots \times B_{\delta_\ell}^{d_\ell}(x^{(\ell)})\right) \lesssim \prod_{i=1}^{\ell} \delta_i^{s-(d-d_i)}.$$

Proof. For technical ease, we proceed using boxes instead of balls, noting that the results are equivalent. Fixing $x \in \mathbb{R}^d$ and $\delta_1, \ldots, \delta_\ell > 0$ with $\delta_j \le \delta_i$ for all $1 \le i \le \ell$, we see that $R = R_{\delta_1}^{d_1}(x^{(1)}) \times \cdots \times R_{\delta_\ell}^{d_\ell}(x^{(\ell)})$ is precisely obtained by stretching $R_{\delta_j}^d(x)$ by a factor of δ_i/δ_j in each hyperplane, so in particular R is contained in $\prod_{i \ne j} \lceil \delta_i/\delta_1 \rceil^{d_i}$ translated copies of $R_{\delta_j}^d(x)$.

Therefore,

$$\mu(R) \lesssim \delta_j^s \prod_{i \ne j} \lceil \delta_i/\delta_j \rceil^{d_i},$$

and the lemma follows. □

Proof of Lemma 2.3. Given the hypotheses of the lemma, we let $s = \dim(E)$, we define $\epsilon > 0$ by

$$4\epsilon = s - \left(d - \frac{\min d_i}{2} + \frac{1}{3}\right),$$

and we let μ be any finite non-negative Borel measure supported on E satisfying

$$\mu\left(B_\delta^d(x)\right) \lesssim \delta^{s-\epsilon} \tag{2.7}$$

for all $x \in \mathbb{R}^d$ and $\delta > 0$. The existence of μ is guaranteed by Frostman's Lemma (see, for example, [7], [13]). We will estimate the integral in (2.5) by iteratively applying Lemma 2.4 to "Fourier slice" measures λ_i on $\mathbb{R}^{d_i}$, defined for fixed $\xi^{(1)}, \ldots, \xi^{(i-1)}, \xi^{(i+1)}, \ldots, \xi^{(\ell)}$ by

$$\widehat{\lambda}_i(\xi^{(i)}) = \widehat{\mu}\left(\xi^{(1)}, \ldots, \xi^{(\ell)}\right).$$

Indeed, the integral in (2.5) is

$$\int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 \int_{\prod_{i=2}^{\ell} S^{d_i-1}} \left(\int_{S^{d_1-1}} |\widehat{\mu}(|\xi^{(1)}|\theta^{(1)}, \ldots, |\xi^{(\ell)}|\theta^{(\ell)})|^2 d\theta^{(1)}\right) d\theta^{(2)} \cdots d\theta^{(\ell)} d\xi$$

$$\lesssim \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 \int_{\prod_{i=2}^{\ell} S^{d_i-1}} |\xi^{(1)}|^{-\frac{d_1+2\alpha_1-2}{4}+\epsilon} \left(\int_{\mathbb{R}^{d_1}} |\widehat{\mu}(\eta^{(1)}, \ldots, |\xi^{(\ell)}|\theta^{(\ell)})|^2 |\eta^{(1)}|^{-d_1+\alpha_1} d\eta^{(1)}\right) d\theta^{(2)} \cdots d\theta^{(\ell)} d\xi$$

$$\vdots$$

$$\lesssim \left(\int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 \prod_{i=1}^{\ell} |\xi^{(i)}|^{-\frac{d_i+2\alpha_i-2}{4}+\epsilon} d\xi\right) \left(\int_{\mathbb{R}^d} |\widehat{\mu}(\eta)|^2 \prod_{i=1}^{\ell} |\eta^{(i)}|^{-d_i+\alpha_i} d\eta\right),$$

provided that $\frac{d_i}{2} \le \alpha_i \le \frac{d_i+2}{2}$ for all $1 \le i \le \ell$. Expressing the integrals from the last line on the space side (using the "Fourier slice" measures and (2.6)), we obtain a constant multiple of

$$\left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \prod_{i=1}^{\ell} |x^{(i)} - y^{(i)}|^{\frac{d_i+2\alpha_i-2}{4}-\epsilon-d_i} d\mu(x) d\mu(y)\right) \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \prod_{i=1}^{\ell} |x^{(i)} - y^{(i)}|^{-\alpha_i} d\mu(x) d\mu(y)\right).$$

By decomposing dyadically into regions where $2^{-j_i} \le |x^{(i)} - y^{(i)}| \le 2^{-j_i+1}$ and then applying Lemma 2.5 and (2.7), we see that convergence of the integrals is implied by convergence of the sums

$$\prod_{i=1}^{\ell} \sum_{j_i=0}^{\infty} 2^{-j_i\left(\frac{d_i+2\alpha_i-2}{4} - d_i + s - (d-d_i) - 2\epsilon\right)} \qquad \text{and} \qquad \prod_{i=1}^{\ell} \sum_{j_i=0}^{\infty} 2^{-j_i(-\alpha_i + s - (d-d_i) - \epsilon)}.$$

Convergence of the former sum is equivalent to $\frac{d_i+2\alpha_i-2}{4} + s - (d-d_i) > d_i + 2\epsilon$ for all $1 \le i \le \ell$. Convergence of the latter sum is equivalent to $\alpha_i < s - (d-d_i) - \epsilon$ for all $1 \le i \le \ell$. Recalling the definition of ϵ and the requirement that $\frac{d_i}{2} \le \alpha_i \le \frac{d_i+2}{2}$ for all $1 \le i \le \ell$, we see that all inequalities can be satisfied by setting

$$\alpha_i = \min\left\{ s - (d-d_i) - 2\epsilon, \frac{d_i+2}{2} \right\}.$$

□

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