MATH 3580 HOMEWORK 1

DUE FRIDAY 9/15

Chapter 1

(1) Express the following complex numbers in rectangular $(x + iy)$ form.

a) $\dfrac{70(4+5i)}{1-3i}$

b) $\dfrac{(4-i)^3}{(1+i)^2}$

c) $(1+i\sqrt{3})^{17}$

(2) Express the following complex numbers in polar $(re^{i\theta})$ form.

a) $6\sqrt{3} - 6i$

b) $-12 - 5i$.

(3) Use Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, as well as the standard angle sum identities

$$\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$$

and

$$\sin(\theta + \phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi),$$

to show that

$$e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}.$$

(4) Either prove or provide a counterexample to the following statements:

a) If $z = re^{i\theta}$ is the polar form of a nonzero complex number, then $\frac{1}{z} = \frac{1}{r}e^{-i\theta}$.

b) If z is a complex number such that z^n is real for infinitely many integers n, then z is real.

c) If $n \in \mathbb{N}$ and z is a nonzero complex number, then z has exactly n n-th roots. However, at most two of those n-th roots are real numbers.

(5) Determine the following collections of roots of the given complex numbers, and plot them in the complex plane. (You do not need to provide the rectangular forms of the roots.)

a) The sixth roots of 64.

b) The eigth roots of $-i$.

c) The cube roots of $-7 + 7i$.

(6) As we have discussed, there are exactly 30 complex numbers satisfying $z^{30} = 1$, which we refer to as the 30-th *roots of unity*. For how many of these roots is it the case that NONE of the smaller powers, namely $z, z^2, z^3, \ldots, z^{29}$, are equal to 1? Such roots are referred to as *primitive* roots of unity. Based on your findings, can you determine a (quick) way, given $n \in \mathbb{N}$, to find the number of primitive n-th roots of unity?

(7) Recall the following basic facts of planar geometry:

- The collection of points in the plane that are a fixed distance away from a fixed point is a circle.
- The collection of points in the plane with a fixed sum of distances from two fixed points is an ellipse.

- The collection of points in the plane that are equidistant from two fixed points is a line.

Using these facts, draw the collections of points in the complex plane that satisfy the following equations or inequalities.

a) $|z - 3 + 2i| \geq 4$

b) $|z - 3i| + |z + 3i| < 10$

c) $|z - 1 - i| = |z + 1 + i|$

(8) Recall that for a set $U \subseteq \mathbb{C}$, an *accumulation point* of U is a point $z \in \mathbb{C}$ such that every neighborhood of z contains a point in U other than z itself. We said in class that a set $U \subseteq \mathbb{C}$ is *closed* if it contains all of its accumulation points. We also said U is closed it its complement $U^C = \mathbb{C} \setminus U$ is open. Prove that those two definitions are equivalent.

Chapter 2

(9) For each of the following functions, separate the function into its real and imaginary parts, writing it as

$$f(x + iy) = u(x, y) + iv(x, y).$$

For example, in class we wrote $f(z) = z^2$ as

$$f(x + iy) = (x^2 - y^2) + i(2xy).$$

a) $f(z) = z^4 - 5$

b) $f(z) = \dfrac{z}{z + 3\bar{z} + 2i}$

c) $f(z) = \dfrac{1}{9 - |z|^2}$

(10) Determine the implied domain of definition for each of the functions in Problem (8).

(11) Decompose the function $f(z) = e^{z^2}$ in the following four ways:

a) Rectangular-to-Rectangular: $f(x+iy) = u(x,y) + iv(x,y)$

b) Rectangular-to-Polar: $f(x+iy) = \rho(x,y)e^{i\phi(x,y)}$

c) Polar-to-Rectangular: $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$

d) Polar-to-Polar: $f(re^{i\theta}) = \rho(r,\theta)e^{i\phi(r,\theta)}$

(12) Determine the image of the given set under the given mapping.

a) Set: $\{re^{i\theta} : 0 \leq r \leq 4,\ 0 \leq \theta \leq \pi/2\}$, Map: $f(z) = z^3$

b) Set: $\{x+iy : 1 \leq x \leq 2,\ 0 \leq y \leq \pi/2\}$, Map: $f(z) = e^{2z}$

c) Set: The imaginary axis $\{iy : y \in \mathbb{R}\}$, Map: $f(z) = z + \bar{z}$

(13) Determine the preimage of the given set under the given mapping.

a) Set: $\{re^{i\theta} : 0 \leq r \leq 4,\ 0 \leq \theta \leq \pi/2\}$, Map: $f(z) = z^3$

b) Set: $\{z : |z| \leq 10\}$, Map: $f(z) = e^{2z}$

c) Set: $\{x+iy : -10 \leq x \leq 6,\ y \in \mathbb{R}\}$, Map: $f(z) = z + \bar{z}$