SETS IN $\mathbb{R}^d$ WITH SLOW-DECAYING DENSITY THAT AVOID AN UNBOUNDED COLLECTION OF DISTANCES

ALEX RICE

ABSTRACT. For any $d \in \mathbb{N}$ and any function $f : (0, \infty) \to [0, 1]$ with $f(R) \to 0$ as $R \to \infty$, we construct a set $A \subseteq \mathbb{R}^d$ and a sequence $R_n \to \infty$ such that $\|x - y\| \neq R_n$ for all $x, y \in A$ and $\mu(A \cap B_{R_n}) \geq f(R_n)\mu(B_{R_n})$ for all $n \in \mathbb{N}$, where B_R is the ball of radius R centered at the origin and μ is Lebesgue measure. This construction exhibits a form of sharpness for a result established independently by Furstenberg-Katznelson-Weiss, Bourgain, and Falconer-Marstrand, and it generalizes to any metric induced by a norm on $\mathbb{R}^d$.

1. Introduction

For $d \in \mathbb{N}$ and $R > 0$, we let B_R denote the standard open ball in $\mathbb{R}^d$ of radius R, centered at the origin, and for $A \subseteq \mathbb{R}^d$, we let A_R denote $A \cap B_R$. Further, we let μ denote Lebesgue measure on $\mathbb{R}^d$, and $\|\cdot\|$ denote the standard Euclidean norm on $\mathbb{R}^d$. We say that $A \subseteq \mathbb{R}^d$ has *positive upper density* if $\limsup_{R\to\infty} \mu(A_R)/\mu(B_R) > 0$. Bourgain [1] showed via harmonic analysis that if $A \subseteq \mathbb{R}^2$ has positive upper density, then A determines all sufficiently large distances, meaning there exists $R_0 = R_0(A)$ such that for all $R > R_0$, there exist $x, y \in A$ with $\|x - y\| = R$. Further, he generalized his proof to establish the analogous result in $\mathbb{R}^d$ with distances replaced by isometric copies of dilates of any d-point configuration that is not contained in a single $(d-2)$-dimensional plane. In his paper, Bourgain alludes to the $d = 2$ result as having been previously established via ergodic theory by Katznelson and Weiss, and this argument was later published by those two authors in joint work with Furstenberg [3]. A short geometric proof of the $d = 2$ case was also found by Falconer and Marstrand [2] around the same time.

Written contrapositively, this common theorem states that if $A \subseteq \mathbb{R}^2$ misses a sequence of distances tending to infinity, then $\lim_{R\to\infty} \mu(A_R)/\mu(B_R) = 0$. One could hope to quantitatively strengthen this conclusion with an estimate of the form $\mu(A_R)/\mu(B_R) \leq f(R)$ for all $R \geq R_0(A)$, where $f(R) \to 0$ independent of A, such as $f(R) = 1/\log R$. Here we establish that such a quantitative improvement is impossible, and hence the aforementioned results are, in a sense, sharp. In particular, given $d \in \mathbb{N}$ and a function $f : (0, \infty) \to [0, 1]$ tending to 0 as $R \to \infty$, we construct a set $A \subseteq \mathbb{R}^d$ and a sequence $R_n \to \infty$ such that $\|x - y\| \neq R_n$ for all $x, y \in A$ and $\mu(A_{R_n}) \geq f(R_n)\mu(B_{R_n})$ for all $n \in \mathbb{N}$. In addition to working in any dimension d, the construction generalizes to any metric induced by a norm on $\mathbb{R}^d$.

2. Motivation

Since the set of distances between integer lattice points in $\mathbb{R}^d$, which we denote by Δ_d, is a fairly sparse, discrete set consisting entirely of square roots of integers, a natural candidate for a dense set $A \subseteq \mathbb{R}^d$ with lots of missing distances is a union of *thickened* lattice points, meaning small balls around lattice points. However, for $d \geq 2$, based on standard results concerning sums of squares, the gaps between consecutive elements of Δ_d tend to 0, hence the full integer lattice, thickened at even a single point, determines all sufficiently large distances. Alternatively, since Δ_d has no limit points, we can certainly thicken a cube $L = [a, b]^d \cap \mathbb{Z}^d$ by a fixed $\epsilon > 0$ and avoid a single distance $R \notin \Delta_d$. The relative density of the thickened cube in $[a, b]^d$ is about ϵ^d, and we can choose L to be as large as we want with respect to ϵ. We iterate this process to form a union of cubes, far enough apart so as to control the interactions between them, that avoids a rapidly growing sequence of distances that fail to occur in the lattice. The crucial detail is that, while the closing gaps between elements of Δ_d force the density of our set to decay to 0, we have control over the relationship between the relative density and the scale of the cube at each step of the construction.

3. Construction for Arbitrary Norms

Recall that $\rho : \mathbb{R}^d \to [0, \infty)$ is called a *norm* if $\rho(x) > 0$ for all $x \neq \vec{0}$, $\rho(\lambda x) = |\lambda|\rho(x)$ for all $\lambda \in \mathbb{R}$ and all $x \in \mathbb{R}^d$ (homogoneity), and $\rho(x + y) \leq \rho(x) + \rho(y)$ for all $x, y \in \mathbb{R}^d$ (triangle inequality). It is a standard fact that all norms on $\mathbb{R}^d$ are equivalent, meaning there exist constants $c_\rho, C_\rho > 0$ such that

$$c_\rho\|x\| \leq \rho(x) \leq C_\rho\|x\| \text{ for all } x \in \mathbb{R}^d. \tag{1}$$

For a norm ρ on $\mathbb{R}^d$, we let $B_R^\rho = \{x \in \mathbb{R}^d : \rho(x) < R\}$, and $A_R^\rho = A \cap B_R^\rho$. Homogeneity of ρ ensures that

$$\mu(B_R^\rho) = \omega_\rho R^d, \tag{2}$$

where $\omega_\rho = \mu(B_1^\rho) \geq c_\rho^d \mu(B_1)$. A norm ρ on $\mathbb{R}^d$ also determines a metric on $\mathbb{R}^d$, by defining the *ρ-distance* between x and y to be $\rho(x - y)$. Our main result is the following.

Theorem 1. *Suppose $f : (0, \infty) \to [0, 1]$, $d \in \mathbb{N}$, and ρ is a norm on $\mathbb{R}^d$. If $f(R) \to 0$ as $R \to \infty$, then there exists $A \subseteq \mathbb{R}^d$ and a sequence of positive numbers $R_n \to \infty$ such that:*

(i) *$\rho(x - y) \neq R_n$ for all $x, y \in A$ and all $n \in \mathbb{N}$.*

(ii) *$\mu(A_{R_n}^\rho) \geq f(R_n)\mu(B_{R_n}^\rho)$ for all $n \in \mathbb{N}$.*

Proof. Fix $d \in \mathbb{N}$, $f : (0, \infty) \to [0, 1]$ with $f(R) \to 0$ as $R \to \infty$, a norm ρ on $\mathbb{R}^d$, and constants $c_\rho, C_\rho > 0$ satisfying (1). Let $1 = R_0 < R_1 < R_2 < \cdots$ and $1 = \epsilon_0 \geq \epsilon_1 \geq \epsilon_2 \geq \cdots$ be any sequences satisfying

(a) $|\rho(x) - R_j| > \epsilon_n$ for all $x \in \mathbb{Z}^d$ and all $1 \leq j \leq n$,

(b) $R_n \geq 100R_{n-1}$, and

(c) $f(R_n) \leq \left(\frac{\epsilon_{n-1}}{16C_\rho}\right)^d$

for all $n \in \mathbb{N}$. We note that property (a) is achievable because, for any interval $[a, b] \subseteq [0, \infty)$,

$$\{x \in \mathbb{Z}^d : \rho(x) \in [a, b]\} \subseteq \{x \in \mathbb{Z}^d : \|x\| \in [a/C_\rho, b/c_\rho]\}$$

is finite, hence $\rho(\mathbb{Z}^d)$ is a discrete set with no limit points.

Let L_n be a translate of $\{0, 1, \ldots, \lfloor R_n/(4C_\rho)\rfloor\}^d$ lying in $B_{R_n/2}^\rho \setminus B_{10R_{n-1}}^\rho$, which exists by (b), let

$$P_n = \{y \in \mathbb{R}_d : \rho(x - y) < \epsilon_{n-1}/4 \text{ for some } x \in L_n\},$$

and let $A = \bigcup_{n=1}^\infty P_n$. For condition (ii), we see that $P_n \subseteq B_{R_n}^\rho$, hence

$$\mu(A_{R_n}^\rho) \geq \mu(P_n) \geq \omega_\rho \left(\frac{\epsilon_{n-1}}{16C_\rho}\right)^d R_n^d \geq f(R_n)\mu(B_{R_n}^\rho),$$

where the second inequality uses that P_n is a disjoint union of at least $[R_n/(4C_\rho)]^d$ ρ-balls of radius $\epsilon_{n-1}/4$, and the third inequality comes from (c) and (2).

Further, we argue inductively that, for each $n \in \mathbb{N}$, $\rho(x - y) \neq R_j$ for all $x, y \in A_{R_n}^\rho$ and all $j \in \mathbb{N}$, which suffices to establish (i). $A_{R_n}^\rho \subseteq B_{R_n/2}^\rho$, so $A_{R_n}^\rho$ certainly has no R_j ρ-distances for $j \geq n$, which in particular establishes the base case $n = 1$. We now fix $n \geq 2$ and suppose $A_{R_{n-1}}^\rho$ has no R_j ρ-distances for all $j \in \mathbb{N}$, and we wish to say the same for $A_{R_n}^\rho = A_{R_{n-1}}^\rho \cup P_n$. Since $A_{R_{n-1}}^\rho$ lies inside $B_{R_{n-1}}^\rho$ and P_n lies outside $B_{10R_{n-1}}^\rho$, the ρ-distance between any point in $A_{R_{n-1}}^\rho$ and any point in P_n is at least $9R_{n-1}$, so it suffices to show that for every $n \in \mathbb{N}$, $\rho(x - y) \neq R_j$ for all $x, y \in P_n$ and all $1 \leq j \leq n - 1$.

By the triangle inequality, the ρ-distances between points in P_n are all within $\epsilon_{n-1}/2$ of ρ-distances between points in L_n. However, by (a), no such distances can equal R_j for $1 \leq j \leq n-1$, and property (i) follows. □

4. Concluding Remarks

It is worth noting that Theorem 1 is only informative for a particular norm if the aforementioned positive result, that sets of positive upper density determine all sufficiently large distances, holds with respect to that norm. For two examples where the result fails, if ρ is the ℓ^∞ or ℓ^1 norm, then B_1^ρ is a $2d$ or 2^d faced polytope, respectively, and $\rho(\mathbb{Z}^d)$ is the set of nonnegative integers, so one can take A to be the full integer lattice thickened by $\epsilon = 1/8$, and A has positive density but misses all half-integer distances. Kolountzakis [4] showed that this type of example is the only obstruction, establishing that the positive result holds for a norm ρ on $\mathbb{R}^d$ provided B_1^ρ is not a polytope, expressly because an analog of the construction from the previous sentence, a thickening of a well-distributed set with a separated collection of ρ-distances, is impossible. It is not known when exactly such a construction is possible in the event that B_1^ρ is a polytope, so the full converse of the Kolountzakis result is still open. Whatever the precise collection of norms may be, Theorem 1 ensures that whenever the positive result holds, it is sharp in the sense discussed in the introduction.

Acknowledgements: The author would like to thank Alex Iosevich for posing this problem at the 2019 Combinatorial and Additive Number Theory (CANT) conference, held at the CUNY Graduate Center in memory of Jean Bourgain. The author would also like to thank Neil Lyall for his helpful discussions.

References

[1] J. Bourgain, *A Szemerédi type theorem for sets of positive density in* $\mathbb{R}^k$, Israel J. Math. 54 (1986), no. 3, 307-316.
[2] K. J. Falconer, J. M. Marstrand, *Plane sets with positive density at infinity contain all large distances*, Bull. London Math. Soc. 18 (1986), no. 5, 471-474.
[3] H. Furstenberg, Y. Katznelson, B. Weiss, *Ergodic theory and configurations in sets of positive density*, Mathematics of Ramsey Theory, Algorithms and Combinatorics, vol. 5 (1990), Springer-Verlag, Berlin, 184-198.
[4] M. Kolountzakis, *Distance sets corresponding to convex bodies*, Geom. Funct. Anal. 14 (2004), no. 4, 734-744.

Department of Mathematics, Millsaps College, Jackson, MS 39210

Email address: `riceaj@millsaps.edu`