2019 Millsaps College
High School Mathematics Competition

Ciphering Round **Solutions**
10 Problems/3 Minutes Each

- All problems are free response, and 10 points are awarded for each correct answer
- The only things allowed out during the round are the pages from this packet, writing utensils, and scratch paper. In particular, **no calculators or electronic devices of any kind are allowed out during the round.**
- Problems will be worked, and then collected, one at a time. **Do not look at the next page in the packet until directed by a proctor to do so.**
- All work during this round must be done as an individual. **No conversation is allowed during this round.**
- Write your name and team on each problem page, and record your answers in the box provided below each problem. **Only what is inside the box will be considered during the scoring process.**

Ciphering Round Problems

(1) Bree is 4 years older than her brother Sam. 30 years ago, Bree was three times as old as Sam. How old is Bree today?

(2) For real numbers a and b, we define the binary operation $a\#b$ by $a\#b = \sqrt{a^2 + b^2}$. Find $((3\#4)\#12)\#84$.

(3) Suppose the polynomial $P(x) = (2x - 1)^6$ is written out in standard form

$$P(x) = a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

Find $64a_6 + 32a_5 + 16a_4 + 8a_3 + 4a_2 + 2a_1 + a_0$.

(4) A basketball team consists of ten players: three sisters from the Smith family, two sisters from the Jones family, and five other women who do not have any sisters on the team. If two players are randomly selected from the team (without replacement), what is the probability that the two players are sisters? Express your answer as a fraction.

(5) A rectangular box has two faces of area 15 square feet apiece, two faces of area 39 square feet apiece, and two faces of area 65 square feet apiece. Measured in cubic feet, what is the volume of the box?

(6) Ten students take a test, and they all receive distinct scores (meaning no repeats) that are integers between 0 and 100, inclusive. What is the greatest possible difference between the mean and median of the ten scores? Express your answer as a decimal.

(7) A regular hexagon H is scaled by a factor of 2, resulting in a new regular hexagon H' with sides twice as long as those of H. Then, the midpoints of adjacent sides of H' are connected to form yet another regular hexagon H''. What is the ratio of the area of H'' to the area of H?

(8) A race car driver completes one lap around a track with an average speed of 90 miles per hour. What must his average speed be on a second lap around the same track in order for his average speed over the two laps to be 120 miles per hour?

(9) When Evan and Joel play ping-pong, Evan wins each game with probability 3/4. Instead of just one game, they decide to play a best-of-three series, meaning they play three games, and whoever wins the most games wins the series. What is the probability that Evan will win the best-of-three series? Express your answer as a fraction.

(10) Find the sum of all real solutions to the equation $2e^{3x} + 9 = 5e^{-3x}$.

Ciphering Round Solutions

(1) Letting B and S denote Bree and Sam's ages, respectively, we have $B = S + 4$ and $B - 30 = 3(S - 30)$. Substituting $S = B - 4$ into the second equation yields $B - 30 = 3B - 102$, so $2B = 72$ and hence $B = \mathbf{36}$.

More informally, one could observe that a four-year difference constituting a $3:1$ age ratio quickly implies that the two siblings former ages were 6 and 2, meaning they are now 36 and 32.

(2) $3\#4 = \sqrt{9+16} = 5$, $5\#12 = \sqrt{5^2+12^2} = 13$, $13\#84 = \sqrt{13^2+84^2} = \mathbf{85}$

(These were rigged to generate Pythagorean triples where the longer leg and hypotenuse are one apart.)

(3) One could certainly expand $P(x)$ and determine each coefficient, but the much faster approach is to observe that the desired expression is $P(2)$, hence the answer is $(2(2)-1)^6 = 3^6 = \mathbf{729}$.

(4) There are $10 \cdot 9$ ways to choose two players in order without replacement. $3 \cdot 2 = 6$ of those ordered pairs are Smith sisters, and $2 \cdot 1 = 2$ of those ordered pairs are Jones sisters, yielding an answer of $8/90 = \mathbf{4/45}$.

(5) Observing that $15 = 3 \cdot 5$, $39 = 3 \cdot 13$, and $65 = 5 \cdot 13$, it must be the case that the dimensions of the box are 3, 5, and 13, hence the volume is $3 \cdot 5 \cdot 13 = \mathbf{195}$ cubic feet.

Alternatively, $lwh = \sqrt{lw \cdot lh \cdot wh}$, so the volume of a box is the square root of the product of its three face areas, hence the volume in this case is $\sqrt{15 \cdot 39 \cdot 65} = 195$.

(6) Suppose the test scores are $0, 1, 2, 3, 4, 5, 97, 98, 99, 100$. In this case, the median is 4.5, while the mean is $(1 + 2 + 3 + 4 + 5 + 97 + 98 + 99 + 100)/10 = 40.9$, so the difference is $\mathbf{36.4}$

None of the top four scores can be increased without creating a repeat, and if any of the top four scores are decreased, the mean will decrease while the median will not, so this change decreases the difference. None of the bottom five scores can be decreased without creating a repeat, and if any of the top five scores are increased, the median will increase more than the mean will increase, so this change decreases the difference.

(7) Suppose H has sidelength 1, so H' has sidelength 2. Half of a side of H'', call it L, is the leg of a right triangle whose hypotenuse is half of a side of H', which has length 1, and the angle opposite L, is half of an interior angle of H', which hence measures 60°. Therefore, L has length $\sqrt{3}/2$, so H'' has sidelength $\sqrt{3}$. Finally, scaling a two-dimensional region by a factor of $\sqrt{3}$ scales its area by $(\sqrt{3})^2 = \mathbf{3}$.

(8) Assume the lap is one mile. At an average of 90mph, he completed the first lap in 40 seconds. To average 120mph on the two laps, he must complete the two laps in one minute. Therefore, he must complete the second lap in 20 seconds, which is an average speed of **180**mph.

More generally, the average speed over two stretches of equal distance is the *harmonic average* of the two average speeds, so this problem amounts to solving the equation $\frac{1}{2}(\frac{1}{90}+\frac{1}{x})=\frac{1}{120}$.

(9) The chance that Evan wins all three games is $(3/4)^3 = 27/64$. Further, each of the three possible orders in which Evan can win two games and lose one game has probability $(3/4)^2(1/4) = 9/64$. Therefore, the total probability that Evan wins the series is $27/64 + 3(9/64) = 54/64 = \mathbf{27/32}$.

(10) Multiplying both sides of the equation by e^{3x} yields $2e^{6x} + 9e^{3x} = 5$, hence

$$2e^{6x} + 9e^{3x} - 5 = (2e^{3x} - 1)(e^{3x} + 5) = 0.$$

Setting the two factors equal to 0 yields $e^{3x} = 1/2$ or $e^{3x} = -5$, the latter of which has no solution and the former of which has one solution $x = \mathbf{ln(1/2)/3}$.