2019 Millsaps College
High School Mathematics Competition

Team Round **Solutions**
5 Problems/60 Minutes

- All problems are free response, and 40 points are awarded for each correct answer.
- The only things allowed out during the round are the pages from this packet, writing utensils, and scratch paper. In particular, **no calculators or electronic devices of any kind are allowed out during the round.**
- This round is collaborative. Teammates are encouraged to work together and communicate during the round, but there should not be any communication between teams.
- This packet contains four copies (one for each teammate) of a single page containing all five problems, followed by a single answer sheet. Record your team's answers on the single answer sheet provided at the back of this packet. **Only the answer sheet should be turned in.** Team members may keep their copy of the problems page.

Team Round Problems

(1) Consider the integer $n = 2019! = (2019)(2018)(2017)\cdots(3)(2)(1)$. How many times must n be cut in half before the resulting integer is odd? For example, if $n/2$ and $n/4$ were even, but $n/8$ were odd (it's not), then the answer would be 3.

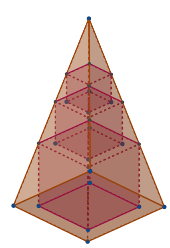
(2) Given a set of numbers, we define the set of *pairwise sums* to be the set of all possible sums of two distinct numbers from the set. For example, the set of pairwise sums of $\{1, 4, 9, 10\}$ is $\{5, 10, 11, 13, 14, 19\}$. If the set of pairwise sums of five numbers is

$$22, 32, 36, 42, 46, 56, 100, 104, 114, 124,$$

list the five numbers in increasing order.

(3) The Ole Miss and Mississippi State basketball teams decide to hold two preseason scrimmages over a period of two weeks, but they want to keep the start times and locations secret. In a secret process, a start time is chosen completely at random out of all possible times during the first week, and then a fair coin is flipped to determine if the game is to be held in Oxford or Starkville. The exact same procedure is then repeated for the second week: a random start time during the second week, and a coin flip for location. After the secret process, information is leaked that at least one of the two games will be held in Oxford with a start time between 3am and 4am on a Wednesday. Knowing only that information, what is the probability that both games will be held in Oxford? Express your answer as a fraction.

(4) Let P be a square pyramid with base side length 3. Suppose an infinite sequence of cubes, the largest having side length 2, is inscribed in P as shown. What is the volume of the portion of P that lies outside of the cubes? Express your answer as a fraction.



(5) Robbie and Julia decide to compete in a coin flipping contest over a weekend. On Saturday, the two competitors independently choose a number of flips to attempt between 1 and 30, inclusive. They repeat the same process on Sunday. The two competitors do not have to flip the same number of coins on the two days, and they do not have to flip the same number of coins as each other. On Saturday, Robbie flips a higher percentage of heads than Julia. On Sunday, Robbie flips a higher percentage of heads than Julia. For the two days combined, $A\%$ of Robbie's flips were heads, and $B\%$ of Julia's flips were heads. What is the least possible value of $A - B$, rounded to the nearest integer?

Team Round Solutions

(1) We need to determine the number of 2's in the prime factorization of 2019!, since this is the number of 2's we must divide away before we are left with an odd number. Each even number $2, 4, 6, \ldots, 2018$, of which there are 1009, contributes a factor of 2. Further, each multiple of 4, up to 2016, of which there are 504, contributes an additional factor of 2. Similarly, the 252 multiples of 8 contribute an additional 2, followed by the 126 multiples of 16, 63 multiples of 32, 31 multiples of 64, 15 multiples of 128, 7 multiples of 256, 3 multiples of 512, and 1 multiple of 1024. Therefore, the total number of 2's is

$$1009 + 504 + 252 + 126 + 63 + 31 + 15 + 7 + 3 + 1 = \mathbf{2011}.$$

For those interested in a generalization, this is a special case of the formula

$$\sum_{k=1}^{\infty} \lfloor n/p^k \rfloor$$

for the number of p's in the prime factorization of $n!$, where $\lfloor \cdot \rfloor$ is the "floor function" that rounds a number down to an integer.

(2) Let the five numbers be $a < b < c < d < e$. The smallest pairwise sum is definitely $a + b$, so $a + b = 22$, and the largest pairwise sum is definitely $d + e$, so $d + e = 124$. Further, if we add all ten pairwise sums, each number appears four times, so

$$a + b + c + d + e = \frac{22 + 32 + 36 + 42 + 46 + 56 + 100 + 104 + 114 + 124}{4} = 169.$$

Combined with our previous observations we have $c = 169 - 22 - 124 = 23$. Finally, the second smallest pairwise sum is $a + c$, so $a + c = a + 23 = 32$, hence $a = 9$ and $b = 13$, while the second largest pairwise sum is $c + e$, so $c + e = 23 + e = 114$, hence $e = 91$ and $d = 33$. Altogether, the final answer is

$$\mathbf{9}, \mathbf{13}, \mathbf{23}, \mathbf{33}, \mathbf{91},$$

which for the older sports fans among us were the jersey numbers for the Chicago Bulls starting lineup during the late 1990s.

(3) If we break each week into hours, there are a total of $24 \cdot 7 = 168$ equally likely hour-long windows for the start time of each game. Combining this with the coin flip for location, we have $2 \cdot 168 = 336$ equally likely (start hour, location) pairs for each game. The leaked information guarantees that at least one of the two pairs is

(3am-4am, Oxford),

but it could either be the first game or the second game, possibly both. If it is the first game, there are 336 possibilities for the second game, 168 of which take place in Oxford. If it is the second game, there are 336 possibilities for the first game, 168 of which take place in Oxford.

However, both our "success" count and our "total" count have doubly counted the event that BOTH games start between 3am and 4am in Oxford. Therefore, the number of possibilities is $336 + 336 - 1 = 671$, and the total number of successes is $168 + 168 - 1 = 335$, for a final answer of

$$\mathbf{335/671}.$$

(4) Consider the pyramid whose base is the top face of the largest cube. This pyramid, and the cubes inscribed inside of it, are perfectly similar to the original arrangement, just scaled by a factor of 2/3 (because the sidelength of the base was 3 before, and it is 2 now). In particular, the 2 units of height lost from the bigger pyramid to the smaller pyramid represent a third of the total height of the larger pyramid, meaning that original height is 6, and hence the total volume of the original pyramid is

$$V_1 = \frac{1}{3}(3)^2(6) = 18.$$

Further, scaling cubes by 2/3 multiplies their volume by $(2/3)^3 = 8/27$. The first cube has volume 8, so the total volume of all the cubes, which we call V_2, satisfies

$$V_2 = 8 + \frac{8}{27}V_2,$$

which yields $V_2 = 216/19$. Therefore, the final answer is

$$18 - \frac{216}{19} = \frac{\mathbf{126}}{\mathbf{19}}.$$

(This solution was written to intentionally avoid the use of infinite series, though they could certainly be used as well.)

(5) This problem is meant to illustrate an extreme version of what is known in statistics as *Simpson's Paradox*, and a key is to shake yourself free of the assumption that the answer must be nonnegative. Suppose Robbie flips one coin on Saturday, and gets one head, while Julia flips 30 coins on Sunday and gets 29 heads. In particular, Robbie flipped a higher percentage of heads on Saturday. Now suppose Robbie flips 30 coins on Sunday, and gets one head, while Julia flips one coin on Sunday and it lands tails. In particular, Robbie flipped a higher percentage of heads on Sunday. However, for the weekend altogether, Robbie flipped $2/31 \approx 6.45\%$ heads, while Julia flipped $29/31 \approx 93.548\%$ heads, so $A - B \approx -87.097$, which rounds to

$$\mathbf{-87}.$$

I will stop short of a completely rigorous proof that this is the minimum value of $A - B$, but note that the counterintuitive "reversal" phenomenon is caused by the discrepancy in Robbie and Julia's sample sizes for each day, so in order to yield the most extreme reversal, we choose the most extreme discrepancy in sample size.