2019 Millsaps College
High School Mathematics Competition

Written Test **Solutions**
20 Problems/90 Minutes

- All problems are multiple choice with five answer choices. 10 points are awarded for each correct answer, 0 points are awarded for each incorrect answer, and 2 points are awarded for each blank answer.

- The only things allowed out during the round are the test packet, writing utensils, and scratch paper. In particular, **no calculators or electronic devices of any kind are allowed out during the round.**

- All work during this round must be done as an individual. **No conversation is allowed during this round.**

- Record all your answers on the answer sheet provided at the back of the test packet. **Only the answer sheet should be turned in.** You may keep the remainder of your test packet.

Written Test Problems

(1) Four tiny towns each have the same population. When the four populations are combined and added to the 1924 students of the local college (who are not counted in the towns' populations), the total number of people is 10,000. How many people live in each town?

(A) 2016 (B) 2017 (C) 2018 (D) 2019 (E) None of these

(2) What is the integer nearest to $\sqrt{2019}$?

(A) 5 (B) 44 (C) 45 (D) 449 (E) None of these

(3) The advertised size of a rectangular television screen is the length of its diagonal. Assuming the shapes are similar, how many 19-inch television screens does it take to cover a single 95-inch television screen?

(A) 5 (B) 10 (C) 15 (D) 25 (E) None of these

(4) How many different "words" can be formed using all eleven letters in MISSISSIPPI? (They don't need to be actual English words, just arrangements of all the letters.)

(A) 4950 (B) 34650 (C) 40320 (D) 39916800 (E) None of these

(5) Austin has money in his pocket, some combination of pennies, nickels, dimes, quarters, \$1 bills, \$5 bills, and \$10 bills. Amazingly, though, he cannot make change for any of a nickel, dime, quarter, \$1 bill, \$5 bill, \$10 bill, or \$20 bill. What is the most money that Austin could have?

(A) \$19.94 (B) \$20.69 (C) \$21.69 (D) \$21.94 (E) None of these

(6) How many of the first 100 positive integers share a common factor (greater than 1) with 30?

(A) 30 (B) 31 (C) 69 (D) 70 (E) None of these

(7) The three vertices of a triangle lie on a circle of radius 1. If one of the interior angles of the triangle measure 45 degrees, what is the length of the side of the triangle that is opposite this angle?

(A) 1 (B) $\sqrt{2}$ (C) $\sqrt{3}$ (D) 2 (E) None of these

(8) Consider a game in which players can only score in increments of 11 points or 5 points. Further, points can only be gained, never taken away, in these increments. What is the largest number of points that is *impossible* to score in this game?

(A) 28 (B) 34 (C) 39 (D) 46 (E) None of these

(9) If a and b are integers, $a \neq b$, and $a^b = b^a$, what is the product of all possible values of $a + b$?

(A) -36 (B) -6 (C) 0 (D) 6 (E) None of these

(10) Suppose Millsaps Syndrome is a condition known to occur in 1 out of 1000 people. A test is developed that is known to be accurate 99% of the time. In other words, if you have Millsaps Syndrome, the test will return positive 99% of the time and negative 1% of the time, and the reverse holds if you do not have Millsaps Syndrome. If a randomly chosen person tests positive for Millsaps Syndrome, what is the probability that they have it, rounded to the nearest percent?

(A) 9% (B) 19% (C) 91% (D) 99% (E) None of these

(11) Find the sum of all real solutions to the equation

$$\ln\left((x^6 - 28x^3 + 28)^{(x^7 - 16x^6 + 63x^5)}\right) = 0.$$

(A) 0 (B) 4 (C) 16 (D) 20 (E) None of these

(12) Consider an $n \times n$ square checkerboard. Jack wishes to move a checker from one corner of the board to the diagonally opposite corner of the board, with his journey satisfying two conditions: the checker can only move vertically or horizontally to an adjacent square (no diagonal moves), and the checker must visit every square on the board *exactly once*. For which value of n is Jack's goal achievable?

(A) 6 (B) 7 (C) 8 (D) All of A, B, and C (E) None of A,B, and C

(13) Daniel is provided with 10 marbles, 5 blue and 5 red, and two empty boxes. First, he must choose how to distribute all 10 marbles into the two boxes, with the only restriction being that each box must have at least one marble. Then, a fair coin is flipped to choose one of the two boxes, and a marble is drawn randomly from the chosen box. Suppose Daniel chooses the distribution that maximizes the probability of the process resulting in the drawing of a blue marble. What is this maximum probability?

(A) 1/2 (B) 7/10 (C) 13/18 (D) 9/10 (E) None of these

(14) Five distinct points in a plane determine ten distinct line segments drawn between the pairs of points, each of which has a length, which measures the distance between the pair of points. However, these ten distances do not need to all be different, there could be repeats. What is the minimum number of distinct distances determined by five points in a plane?

(A) 2 (B) 3 (C) 5 (D) 7 (E) None of these

(15) Five distinct points in a plane determine ten distinct line segments drawn between the pairs of points, each of which has a midpoint. However, these ten midpoints do not need to all be different, there could be repeats. What is the minimum number of distinct midpoints determined by five points in a plane?

(A) 2 (B) 3 (C) 5 (D) 7 (E) None of these

(16) A fancy glass has the shape of an inverted (point at the bottom) right circular cone. Suppose the glass is filled with water to a level of half the height of the glass. If V_1 is the volume of water in the glass, and V_2 is the volume of the remaining empty space in the glass, what is V_2/V_1?

(A) 1 (B) 3 (C) 4 (D) 8 (E) None of these

(17) 50 doors, numbered 1 through 50, are all open. Person #1 walks by and draws a blue dot on every open door. Person #2 walks by and draws a red dot on every other open door (starting with door 2). After that, for each $3 \leq n \leq 50$, Person $\#n$ comes to every n-th door and executes the following procedure: if there is only a blue dot, they draw a red dot; if there is a red dot, but the door is open, they close the door; if the door is already closed, they leave it closed. After the entire process concludes, how many of the original 50 doors remain open?

(A) 15 (B) 16 (C) 25 (D) 26 (E) None of these

(18) 1000 doors, numbered 1 through 1000, are all open. Person #1 walks by and closes every door. Person #2 walks by and re-opens every other door (starting with door 2). After that, for each $3 \leq n \leq 1000$, Person $\#n$ comes to every n-th door and executes the following procedure: if the door is open, they close it; if the door is closed, they open it. After the entire process concludes, how many of the original 1000 doors are open?

(A) 1 (B) 11 (C) 21 (D) 31 (E) None of these

(19) Five party guests each hang their coat in the same hall closet. The power goes out during the party, so in order to leave the house and endure the cold, bitter evening, each guest is force to grab a coat from the closet at random, without replacement. What is the probability that exactly one of the five guests draws their own coat?

(A) 1/5 (B) 1/4 (C) 11/30 (D) 3/8 (E) None of these

(20) A buffet line has a total of six items. A group of people passes through the buffet line, and each person is free to add as many or as few of the six items to their plate as they please. After everyone makes their selections, someone makes a fascinating observation: no two plates have the exact same collection of items, but every pair of plates has at least one item in common. What is the maximum number of people in the group?

(A) 8 (B) 16 (C) 32 (D) 64 (E) None of these

Written Test Solutions

(1) Letting P denote the common population of the towns, we have $4P + 1924 = 10000$, which yields $P = 2019$. **(D)**

(2) $44^2 = 1936$, while $45^2 = 2025$, so $\sqrt{2019}$ is between 44 and 45 but closer to 45. **(C)**

(3) Scaling a rectangle by a factor of 5 scales both its length and width by 5, hence scaling its area by 25. **(D)**

(4) If we distinguish between all 11 letters, then there are 11! possible orderings. However, we do not want to distinguish between the four S's, four I's, or two P's, so we need to divide to account for the 4!, 4!, and 2! ways that they can be put in order, yielding a final count of $11!/(4!4!2!) = 34650$. **(B)**

(5) Austin can have at most four pennies, but the next choice is a bit more subtle. He clearly cannot have two nickels, but also, if he takes one nickel, then he can only have one dime, since two dimes and a nickel make a quarter. However, if he has no nickels, then he can have up to nine dimes without making change for anything. However, if he has nine dimes he can only have one quarter, or alternatively he could have only four dimes and three quarters, which turns out to be equivalent. After that, everything is straight forward, he can have four \$1's, one \$5, and one \$10, for a grand total of $.04 + .4 + .75 + 4 + 5 + 10 =$\$20.19. **(E) (The original solution had an error in which Austin had nine dimes *and* three quarter for a total of \$20.69, but alas, five dimes and two quarters make \$1. The problem was originally designed to have \$20.19 as an answer, but I guess I forgot that by the time I picked the answer choices. Such is life. . . Both B and E were accepted as correct responses.)**

(6) Let's examine the complement. Between 1 and 30, the positive integers that share no common factors (greater than 1) with 30 are those which fail to be divisible by 2, 3, or 5. The proportion of such integers is $(1/2)(2/3)(4/5) = 8/30$. Then, everything cycles, so there are 8 between 1 and 30, 8 between 31 and 60, and 8 between 61 and 90. Examining the integers between 91 and 100 separately, we see that there are only two more qualifying integers, 91 and 97. Therefore, the complement has total size $8 \cdot 3 + 2 = 26$, and hence the answer is 74. **(E)** (The 8 out of 30 calculation is a special case of *Euler's totient function* $\phi(n)$)

(7) Consider the arc opposite the $45°$ angle. It is a fact that the central angle determined by that arc has twice the measure of the $45°$ angle, hence the side length in question is the hypotenuse of a right triangle whose legs are both radii of the circle, which have length 1. In particular, the length of the side is $\sqrt{2}$. **(B)**

(8) We first see that 39 is impossible, since none of $0, 11, 22, 33$ are a multiple of 5 away from 39. Further, we see that we can get all larger numbers by considering their remainder when divided by 5. By adding multiples of 5 to 0, we get all multiples of 5; by adding multiples of 5 to 11, we get all scores that are at least 11 and have a remainder of 1; by adding multiples of 5 to 22, we get all scores that are at least 22 and have a remainder of 2; by adding multiples of 5 to 33, we get all scores that are at least 33 and have a remainder of 3; finally, by adding multiples of 5 to 44, we get all scores that are at least 44 and have a remainder of 4. This fills out all scores that are greater than 39. **(C)** (This is a special case of the *Frobenius number*, which says that if the increments are a and b, and a and b have no common factors, then the largest impossible score is $ab - a - b$.)

(9) The only possibilities for a and b are 4 and 2 or -4 and -2 (in either order), yielding the answer $(6)(-6) = -36$. Certainly these choices qualify, as both expressions yield 16 and $1/16$, respectively, but to see that there are no others, we start by assuming $0 < a < b$. Since $a \neq b$, neither can be 1. We can rearrange the given equation to $\left(\frac{b}{a}\right)^a = a^{b-a}$. Since the right side is an integer greater than 1, so is the left side, and hence so is b/a. In other words, $b = ka$ for some integer $k \geq 2$. This yields the equation $k^a = a^{(k-1)a}$, and hence $a^{k-1} = k$. If $a = 3$ and $k = 2$ or $a = 2$ and $k = 3$, then $a^{k-1} > k$, and increasing either variable just increases this discrepancy, so the only option is $a = k = 2$, yielding $b = 4$. The fact that 2 and 4 are both even means that we also get the negative integer solution. **(A)**

(10) To make the numbers work out nicely, suppose there are a million people. Since one out of 1000 have Millsaps Syndrome, that means 1000 people are affected. Of those 1000 people, we expect 99%, or 990, to test positive. Out of the other 999,000 people, we expect 1%, or 9990, to test positive. In this case, $9990 + 990 = 10089$ people test positive, of which only 990 have Millsaps Syndrome, yielding a probability of $990/10089 \approx 9\%$. **(A)** (One could also approach this using the formula for conditional probability. The fact that the answer is lower than intuition might suggest is an example of the *Prosecutor's Fallacy*.)

(11) Pulling the exponent to the front we have $(x^7 - 16x^6 + 63x^5)\ln(x^6 - 28x^3 + 28) = 0$, meaning that either $x^7 - 16x^6 + 63x^5 = 0$ or $x^6 - 28x^3 + 28 = 1$. The former equation yields $x^5(x^2 - 16x + 63) = x^5(x-7)(x-9) = 0$, so $x = 0, 7$ or 9. The latter equation yields $x^6 - 28x^3 + 27 = (x^3 - 1)(x^3 - 27) = 0$, so $x = 1$ or $x = 3$. Therefore, the sum of all real solutions is $0 + 7 + 9 + 1 + 3 = 20$. **(D)**

(12) Suppose the squares of the checkerboard are colored black and white in the traditional alternating way. Suppose Jack starts in a corner colored black. His first move will take him to a white square, his second move will take him to a black square, etc. In fact, every odd numbered move will take him to a white square, and every even numbered move will take him to a black square. If he achieves his goal, he will enter the diagonally opposite corner on move $n^2 - 1$, but that corner is also black, so $n^2 - 1$ must be even. This leaves only $n = 7$ as a possibility. To see that he can achieve his goal with $n = 7$, consider "snaking" up and down the columns. **(B)**

(13) If Daniel puts the same number of red marbles as blue marbles in each box, his win probability is 1/2. Can he beat that? If he does anything else, then he is at a disadvantage, in that there are less than half blue marbles, in exactly one of the two boxes. With that in mind, he should try to choose an arrangement that simultaneously minimizes his disadvantage in the "bad" box while maximizing his advantage in the other box. A strategy that achieves both goals to the extreme is to put one blue marble in one box, and the other nine marbles in the other box. In this case, the probability that the process ends with a blue marble is $(1/2)(1) + (1/2)(4/9) = 13/18$. **(C)**

(14) We can see that two distances is possible by considering the vertices of a regular pentagon. By rotational symmetry, there are only two distances, the distance between adjacent vertices and the distance between nonadjacent vertices. To see that one distance is impossible, fix any two points, and call the distance between them d. Any additional points that are distance d from both points lie on the intersection of the two circles of radius d centered at the original two points. Since the two circles have at most two intersection points, we cannot add three additional points, so it is impossible for five points to determine a single distance. In other words, the minimum number of distances determined by five points in a plane is two. **(A)** (The more general question of the minimum number of distances determined by n points in a plane is the famous *Erdős Distance Problem.*)

(15) Consider five equally space points on a line. For simplicity, assume they have coordinates $(0, 0), (2, 0), \ldots, (8, 0)$. The midpoints between pairs of these points have the form $(k, 0)$ where $1 \leq k \leq 7$ is an integer, and in particular there are only 7 distinct midpoints. To see that this is indeed the minimum number, consider any five points, fix the pair of points, call them P and Q, that are the furthest apart, and call their distance d. The four distinct midpoints determined by pairs involving P all lie on or inside the circle of radius $d/2$ centered at P. Similarly, the four distinct midpoints determined by pairs involving Q lie on or inside the circle of radius $d/2$ centered at Q. The regions on or inside these two circles only overlap at a single point, the midpoint between P and Q, so therefore the number of total distinct midpoints is at least $4+4-1 = 7$. **(D)** (This solution generalizes to show that the minimum number of distinct midpoints determined by n points in a plane is $2n - 3$.)

(16) The water forms a cone that is similar to the glass, scaled by a factor of $1/2$. Since the object is three-dimensional, the volume is scaled by a factor of $(1/2)^3 = 1/8$. In other words, $1/8$ of the glass's volume is filled with water, while $7/8$ of the glass is not, so the ratio of the two volumes is 7. **(E)**

(17) The open doors are numbered by positive integers with less than or equal to two factors. A positive integer has this property if and only if it is either 1 or a prime number. In particular, the open doors are numbered

$$1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47,$$

for a total of 16. **(B)**

(18) The open doors are numbered by positive integers with an *even* number of positive factors. Having an odd number of factors is a rare property, since divisors of a positive integer n naturally come in pairs $d, n/d$. The total number of factors will only be odd if there is a divisor d that is paired with itself, namely $d = n/d$, hence $n = d^2$. In other words, the only positive integers with an odd number of factors are the perfect squares. One can also see this using a standard formula for the number of factors in terms of prime factorization. Since $31^2 = 961$ and $32^2 = 1024$, the closed doors at the end of the process are numbered $1^2, 2^2, \cdots, 31^2$, and in particular there are 31 of them. Therefore, there are $1000 - 31 = 969$ open doors. **(E) (The intention of this problem was for the open doors to be perfect squares and for the answer to be 31, but the doors were supposed to start out closed and I wrote it incorrectly. Both D and E were accepted as correct responses.)**

(19) There are $5! = 120$ total ways that the five coats could be chosen by the five guests. In order for an arrangement to meet the condition, we would need one person to get their own coat (5 choices), and the remaining four people to rearrange their coats in a way that nobody gets their own. Since there is no overlap, the number of qualifying arrangements is $5D$, where D is the number of ways that 4 things can be rearranged such that nothing is in its original place (called a *derangement*). Using either a brute force search, the inclusion-exclusion principle, or any other method, one finds that 9 out of the 24 possible orderings of 4 objects are in fact derangements, so $D = 9$, the number of qualifying arrangements is 45, and the final answer is $45/120 = 3/8$. **(D)**

(20) Think of a plate as a subset of $\{1, 2, 3, 4, 5, 6\}$, where the number represent the six available items. The total number of choices for a plate is $2^6 = 64$, because you can either choose or omit item 1, choose or omit item 2, etc. Further, these 64 plate choices can be paired up by pairing a plate with its "complement". In other words, pair the plate $\{1, 3\}$ with the plate $\{2, 4, 5, 6\}$. Since the plates in a complementary pair have no item in common, the given condition that each pair of plates in the group has at least one item in common means that the group can contain at most one plate out of every complementary pair. In particular, the total number of people in the group is at most half the total number of plate choices, which is 32. To see that 32 is indeed possible, suppose every member of the group picked item 1, and then every distinct subset of $\{2, 3, 4, 5, 6\}$ was selected by one member of the group. This group satisfies the conditions, and the number of members is the number of subsets of $\{2, 3, 4, 5, 6\}$, which is $2^5 = 32$. **(C)**