2021 Millsaps College
High School Mathematics Competition

Ciphering Round **Solutions**
10 Problems/3 Minutes Each

- All problems are free response, and 10 points are awarded for each correct answer
- The only things allowed out during the round are the pages from this packet, writing utensils, and scratch paper. In particular, **no calculators or electronic devices of any kind are allowed out during the round.**
- Problems will be worked, and then collected, one at a time. **Do not look at the next page in the packet until directed by a proctor to do so.**
- All work during this round must be done as an individual. **No conversation is allowed during this round.**

Ciphering Round Problems

(1) Cauchy and Magnus are adorable cats. Both cats are less than 20 years old, and Cauchy is at least three years older than Magnus. The product of their ages (in years, rounded down to a whole number as usual) is a two-digit number. Subject to these conditions, the product of their ages is a large as possible. How old is Cauchy?

(2) Chandler and Simba are adorable cats. At 9:00 am, Chandler is 30 miles due north of home, and begins to walk directly toward home at a rate of 3 miles per hour. At the same time, Simba is 3 miles due east of home and begins to walk directly away from home at a rate of 2 miles per hour. What is the distance between the two cats at 11:00 am?

(3) The sticker price of a t-shirt has the property that double the sticker price is \$17 more than half the sticker price. Including sales tax, which is less than \$1, the total cost (sticker price plus sales tax) of the t-shirt comes to a whole number of dollars. What is this total cost?

(4) What is the least positive integer n such that $n!$ is divisible by 729?

(5) What is the product of all real solutions to the equation $x^4 = 15 + 16x^{-4}$?

(6) A bag contains 38 black marbles, 24 red marbles, and 1 gold marble. Jack selects one marble from the bag at random, replaces the selected marble back into the bag, then again selects a marble from the bag at random. Let p be the probability that both of Jack's selections are the same color. If p is written as a fraction in lowest terms, what is the numerator of p?

(7) What is the product of all real solutions to the equation $\cos^2(x) = \sin^2(x)$ with $-\pi < x \leq \pi$? Provide an exact answer in terms of π.

(8) In a room with 30 people, everybody says their birthday out loud, and if two people share the same birthday they high five. What is the expected value of the total number of high fives? Neglect leap years, and assume each birthday is equally likely. Express your answer as a fraction in lowest terms.

(9) What is the domain of the function $f(x) = \ln(\ln(\ln(x)))$, where ln denotes the natural logarithm? Express your answer in interval notation.

(10) Leo is a yellow lab who really loves treats. He lives in a neighborhood with houses numbered

$$1, 2, 3, 5, 8, 13, 21, 34.$$

The distance separating a pair of houses, in blocks, is equal to the distance between the two house numbers. Leo begins by selecting one house to visit, at which point a neighborhood group text thread is initiated to track Leo's comings and goings. After that, Leo can travel to any of the other houses, and the residents of the new house will give him a number of treats that is equal to the distance between their house and the previous house that Leo visited. There is one catch: Leo can visit each house at most once. These people aren't *made* of treats! What is the maximum number of treats that Leo can earn on his journey?

Ciphering Round Solutions

(1) The only way to write 99 as a product of two positive integers under 20 is $9 \cdot 11$, which does not meet the condition that Cauchy is at least three years older. However, $98 = 7 \cdot 14$ satisfies all conditions, and since Cauchy is the older cat, she is **14** years old (but she still acts like a kitten most of the time).

(2) At 11am, Chandler will be 24 miles north of home, while Simba will be 7 miles east of home. The distance D between the cats is the hypotenuse of a right triangle with legs 24 and 7, so $D = \sqrt{24^2 + 7^2} = \mathbf{25}$ by the Pythagorean theorem.

(3) The sticker price x satisfies $2x = x/2 + 17$, hence $3x/2 = 17$, so $x = 34/3$, which is between 11 and 12. Therefore, the condition that the sales tax is under a dollar and makes the total cost a whole number implies that the total cost is $**12**.

(4) Since $729 = 3^6$, we need $n!$ to have at least 6 factors of 3 in its prime factorization. Each multiple of 3 between 1 and n, inclusive, contributes at least one factor of 3 to $n!$, and each multiple of 9 contributes an additional factor. In particular, 15! gets exactly one factor of 3 from $3, 6, 12$ and 15, plus two factors of 3 from 9, for a total of exactly 6, while all smaller factorials have at most 5. Therefore, the answer is $n = \mathbf{15}$.

(5) The expression on the right hand side is undefined for $x = 0$, so we can assume $x \neq 0$ and multiply both sides by x^4, yielding $x^8 = 15x^4 + 16$. Moving everything to one side and factoring we get $x^8 - 15x^4 - 16 = (x^4 - 16)(x^4 + 1) = 0$. The second factor is never 0 for real x, while the first factor is 0 for $x = \pm 2$, so the final answer is **-4**.

(6) The total number of marbles is 63, so the total number of pairs of draws without replacement is 63^2. The total number of draws of two black marbles is 38^2, while the analogous totals for red and gold are 24^2 and 1, respectively. Therefore, $p = (38^2 + 24^2 + 1^2)/63^2 = 2021/63^2$. Since 2021 is not divisible by 3 or 7, this fraction is already in lowest terms, and the answer is **2021**.

(7) The equation holds if and only if $\cos(x) = \pm\sin(x)$, which we can think of as the points on the unit circle intersecting with the lines $y = x$ or $y = -x$. These are in fact the angles with reference angle $\pi/4$ in each quadrant, which restricted to the given range are $x = -3\pi/4, -\pi/4, \pi/4, 3\pi/4$. Therefore, the answer is $\mathbf{9\pi^4/256}$.

(8) There are $\binom{30}{2} = (30)(29)/2 = 435$ pairs of people, each of which has probability $1/365$ of sharing a birthday. By additivity of expected value (regardless of dependence), the expected value of the number of pairs with shared birthdays is $435/365 = \mathbf{87/73}$.

(9) The function $f(x) = \ln(\ln(\ln(x)))$ is defined if and only if $\ln(\ln(x)) > 0$, which holds if and only if $\ln(x) > 1$, which holds if and only if $x > e$. Therefore, the domain is (e, ∞).

(10) Assume without loss of generality that the houses are all in a straight line (in fact, by the triangle inequality, they would have to be). It makes sense that Leo should do a lot of swinging back and forth from end to end of the neighborhood, but where should he start? Some trial and error reveals that it is best to start in the middle, either at house 5 or 8, go to the opposite endpoint, and then swing back and forth as far as possible. This yields the two sequence $(5, 34, 1, 21, 2, 13, 3, 8)$ and $(8, 1, 34, 2, 21, 3, 13, 5)$, each of which yield **127** treats for Leo. To prove more rigorously that this is optimal, one can consider the maximum number of times that Leo can traverse the line segment connecting each pair of neighboring houses, and in fact the two sequences listed precisely achieve that maximum on *every* segment.