2021 Millsaps College
High School Mathematics Competition

Team Round **Solutions**
5 Problems/60 Minutes

- All problems are free response, and 40 points are awarded for each correct answer.
- The only things allowed out during the round are the pages from this packet, writing utensils, and scratch paper. In particular, **no calculators or electronic devices of any kind are allowed out during the round.**
- This round is collaborative. Teammates are encouraged to work together and communicate during the round, but there should not be any communication between teams.

Team Round Problems

(1) While on a trip to Birmingham for a Millsaps recruiting event, your humble question writer noticed his odometer cycle through the mileages 111222, 111333, and 111444, inspiring the following question:

Suppose a six-digit positive integer is called *compatible* if the first three digits, as a three-digit number, divide evenly into the last three digits, as a three-digit number. For example. 111222, 111333, and 111444 are all compatible numbers, as is 341682. To clarify, a number must be a true six-digit number to qualify, so something like 92184 would not count. How many compatible six-digit positive integers are there?

(2) The Decatur High School Logic Club used to be small and exclusive, but it has recently expanded out of control, to 97 members. Anna, the club's president and senior member, hatches a plan. The club will (repeatedly if necessary) vote on whether to (further) reduce its size. In each round of voting, a strict majority (meaning more than half) of "yes" votes results in the immediate ejection of the newest club member, which leads to another round of voting, and so on. If at any point half or more of the surviving members vote "no", the session is terminated and the club membership remains as it currently stands. Suppose that each member places the highest priority on personally remaining on the board, but otherwise agrees that the smaller the club, the better. Also, since it is a logic club, assume that each member is perfectly logical. To what size will this protocol reduce the club?

(3) In the voting for the 2004-05 NBA Most Valuable Player, winner Steve Nash received 65 first place votes, compared to runner-up Shaquille O'Neal's 58. Suppose that these 123 first-place votes (and no others) are counted in a random order. What is the probability that, other than the initial tie of 0-0, Nash remains in the lead for the *entire* counting process?

(4) Many poorly-worded intelligence tests, online quizzes, etc., have questions of the form:

Find the next two numbers in the pattern: $1, 3, 9, 27, \ldots$

In fact, these four numbers could be the beginning of uncountably many sequences, so additional assumptions must be made to extrapolate the "pattern". The natural inclination is to identify that these four terms follow the rule that the next term is triple the previous, yielding a response of $81,\ 243$. But what if we made a different assumption?

Assume that the four terms above are given by a minimal degree polynomial sequence, meaning $P(1) = 1$, $P(2) = 3$, $P(3) = 9$, and $P(4) = 27$, where P is a polynomial function with the smallest possible degree. What are the next two numbers in the pattern, in other words $P(5)$ and $P(6)$?

(5) Suppose n members of a math department conduct a secret gift exchange, in which each person randomly selects three of their colleagues (not including themselves) to whom to give a gift. We refer to the resulting outcome as *perfectly mutual* if every department member receives gifts from the exact same three people to whom they give gifts, and nobody else. To clarify, in a perfectly mutual outcome, each department member both gives and receives exactly three gifts, and person A receives a gift from person B if and only if person B receives a gift from person A.

Let p denote the probability of a perfectly mutual outcome when $n = 6$, and let q denote the probability of a perfectly mutual outcome when $n = 7$. **For 20 points each**, find p and q.

Team Round Solutions

(1) We break the numbers into groups based on how many possible multiples of the first three digits are still three-digit numbers. For example, if the first three digits like between 100 and 111, inclusive, then there are 9 possible multiples for the last three digits, yielding a total of $12 \cdot 9 = 108$ compatible numbers from this group. Continuing in this manner, 112 to 124 yields $13 \cdot 8 = 104$, then 125 to 142 yields $18 \cdot 7 = 126$, then 143 to 166 yields $24 \cdot 6 = 144$, then 167 to 199 yields $33 \cdot 5 = 165$, then 200 to 249 yield $50 \cdot 4 = 200$, then 250 to 333 yield $84 \cdot 3 = 252$, then 334 to 499 yield $166 \cdot 2 = 332$, then 500 to 999 yield 500. Therefore, at long last, the final answer is $108 + 104 + 126 + 144 + 165 + 200 + 252 + 332 + 500 = \mathbf{1931}$.

(2) Consider the more general question with n board members. If $n = 2$, the senior member votes yes, while the junior member votes no, so we stay with 2. If $n = 3$, both the senior member and middle member know that, by the previous case, their spots are safe if the board is reduced to 2, so they vote yes, and the third person is eliminated. If $n = 4$, member 4 (numbered by decreasing seniority) votes no, and further, member 3 knows that, by the previous case, if member 4 is eliminated, member 3 will also be eliminated in the next round, so member 3 votes no, and we have a tie, so we stick with 4 members. Continuing this pattern, we see that if the number of board members is a power of 2, then that number of members is "stable" in that it will result in a tie vote and an end to the reductions. However, every non-power of 2 is "unstable", in that everyone up to the previous power of 2 (which is more than half) of the members will vote yes, resulting in a reduction. This can be framed more rigorously using mathematical induction. In this particular case of $n = 97$, the first stable number that is reached is **64**.

(3) This is an example of what is known as *Betrand's ballot theorem*. The slickest solution is as follows: suppose there are M votes for candidate A and N votes for candidate B with $M > N$, and choose a random ordering of the votes by randomly passing them around a circle, and then randomly selecting where to start counting. The claim is that no matter how you pass the votes around in a circle, the chance of your random selection of starting point yielding an admissable order (one where the winner stays ahead the whole time) is fixed. Indeed, you cannot start the count at any of the N votes for Candidate B, and you can also, for each candidate B vote, eliminate the first non-eliminated candidate A vote that precedes it, as starting at that vote would result in a tie at some point. In particular, $2N$ of the $M+N$ starting points have been eliminated, while all that remain are good, so the probability of a good selection is $(M-N)/(M+N)$. In this particular case, the answer is $(65-58)/(65+58) = \mathbf{7/123}$.

(4) There are a number of valid approaches, but the quickest is to use the fact that a polynomial sequence is defined by the fact that after some number of "difference operations", the sequence is constant. For example, a linear sequence has constant differences, a quadratic sequence has constant difference of differences, a cubic sequence has constant difference of differences of differences, etc. For calculus students, this is a discrete analog of the fact that n-th derivative of a degree n polynomial is

constant. In this case, taking difference operations of $1, 3, 9, 27$ yields $2, 6, 18$, then $4, 12$, then 8. So, to make this a minimal degree polynomial sequence, we just extend that last list of differences to be constant $8, 8, 8, 8 \ldots$, which extends the next level up to $4, 12, 20, 28, \ldots$, which extends the next level up to $2, 6, 18, 38, 66, \ldots$, which extends the original sequence to $1, 3, 9, 27, 65, 131$, so the answers are $P(5) = \mathbf{65}$ and $P(6) = \mathbf{131}$.

(5) First we consider $n = 6$, and we note that a person randomly selecting three colleagues to whom to give gifts is completely equivalent to selecting two colleagues to whom to *not* give gifts, so the problem is identical if we replace "three" in the gift giving procedure with "two". This is not strictly necessary but does simplify the reasoning. Choose any particular person and label them person 1. Since 1 must select some two colleagues, we may as well do the rest of our labeling based on person 1's selections, and assume without loss of generality that 1 draws 2 and 3. Under that restriction, the sample space is now determined by five random selections of two out of five colleagues, so the total number of possibilities is $\binom{5}{2}^5 = 10^5 = 100000$. Now we must count the "successes" under the restriction that 1 chooses 2 and 3. The simplest approach is to model the situation with a graph: six vertices representing the people, and a pair of vertices connected with an edge if and only if *both* vertices draw each other in the gift exchange. In this context,, a "success" is a graph on six vertices where each vertex has exactly two edges coming out of it, under the restriction that vertex 1 connects to 2 and 3. We consider two cases:

Case 1: 2 connects to 3. In this case, the first three vertices are now fully spoken for, and vertices 4, 5, and 6 must fulfill the condition within themselves, forcing each of those three vertices to connect to the other two. The resulting two picture is two disjoint triangles, and this is the only qualifying outcome in this case.

Case 2: 2 connects to 4, 5, or 6. In this case, we can follow the "chain" of connections until we finally connect back to 3. If that ever concludes before all vertices have been accounted for, for example if 2 connects to 4 and then 4 connects to 3, then some subset of either four or five vertices will be fully spoken for amongst themselves, leaving one or two vertices disconnected and unable to fulfill the condition. Therefore, in this case, it must be that the chain of connection runs through all of vertices 4, 5, and 6 before returning to 3. Since this chain can run through the three vertices in any order, there are $3! = 6$ such successes.

Therefore, the total number of successes in this restricted sample space is 7, hence $p = \mathbf{7/100000} = \mathbf{.00007}$. For $n = 7$, the complement trick does not help, since each person has six colleagues, so selecting three for gifting is equivalent to selecting three for nongifting. However, if we again model the situation with a graph, this time on seven vertices, desiring each vertex to have three edges (or *degree* 3), we encounter a problem. Every edge connects two vertices, so the *total degree* must be an even number. But if there are seven vertices, each of degree 3, then the total degree is 21, which is impossible. In other words, $q = \mathbf{0}$.