2021 Millsaps College
High School Mathematics Competition

Written Test **Solutions**
20 Problems/90 Minutes

- All problems are multiple choice with five answer choices. 10 points are awarded for each correct answer, 0 points are awarded for each incorrect answer, and 2 points are awarded for each blank answer.

- The only things allowed out during the round are the test packet, writing utensils, and scratch paper. In particular, **no calculators or electronic devices of any kind are allowed out during the round.**

- All work during this round must be done as an individual. **No conversation is allowed during this round.**

Written Test Problems

(1) What is the integer closest to $\sqrt[3]{2021}$?

(A) 12 (B) 13 (C) 44 (D) 45 (E) None of these

(2) Austin is Ian's older brother. The difference between their ages (rounded down to a whole number of years, as usual) is also the greatest common divisor of their ages. The product of their ages is 54. How old is Austin?

(A) 3 (B) 18 (C) 6 (D) 9 (E) None of these

(3) What is the sum of the prime divisors of 2021?

(A) 90 (B) 91 (C) 2021 (D) 2022 (E) None of these

(4) An 11-inch tall bronze statue of Freddie Freeman is created as a model of a statue to be erected outside of Truist Park in Atlanta. The real statue will be exactly the same shape and proportions as the model, will be made out of the exact same type of bronze, but will be 77 inches tall, just like Freddie. If the model statue weighs 15 pounds, how many pounds will the real statue weigh?

(A) 105 (B) 735 (C) 1155 (D) 5145 (E) None of these

(5) What is the units digit of $7^{7^{7^{7^{7}}}}$?

(A) 7 (B) 9 (C) 3 (D) 1 (E) None of these

(6) What is the sum of all real solutions to the equation $x^7 + 5x^3 = 8x^4 + 40$?

(A) 0 (B) 8 (C) -8 (D) 40 (E) None of these

(7) Which of the following expressions is equal to $(\log_2 3)(\log_3 4)(\log_4 5)\cdots(\log_{2020} 2021)$?

(A) $\log_{2021} 2$ (B) $\log_2 2021$ (C) $\log_{2020!}(2021!/2)$ (D) 0 (E) None of these

(8) Suppose $A = (2,1)$, $B = (17,13)$, and $C = (10,1)$ are points in the xy-plane. What is the area, in square units, of the triangle ABC?

(A) 24 (B) 48 (C) 64 (D) 96 (E) None of these

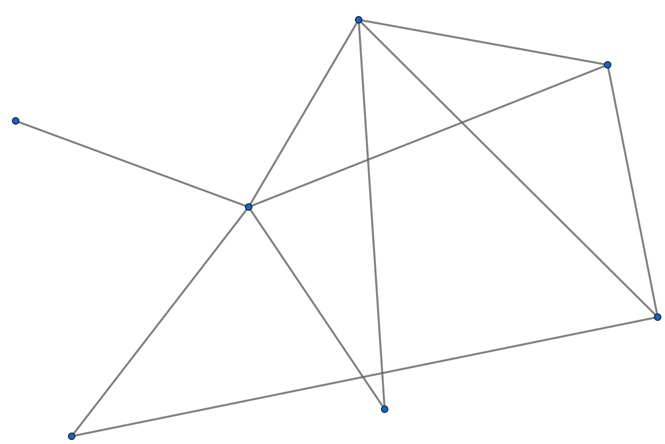
(9) Suppose $A = (2,1)$, $B = (14,6)$, and $C = (16,35)$ are points in the xy-plane. What is the area, in square units, of the triangle ABC?

(A) 169 (B) 144 (C) 328 (D) 288 (E) None of these

(10) What is the range of the function $g(x) = e^{-(x^2+6x+11)}$?

(A) $(0, 1/e^2]$ (B) $(0, e-2]$ (C) $(0,\infty)$ (D) $[1/e^2, \infty)$ (E) None of these

(11) In the diagram below, each dot (vertex) represents a person, and each pair of vertices are connected with a line segment (edge) if and only if the two people are friends. For each vertex v, let $d(v)$ denote the number of friends of v. What is the average value of $d(v)$ for the seven people, rounded to the nearest tenth?



(A) 2.6 (B) 2.7 (C) 2.8 (D) 2.9 (E) None of these

(12) Use the same diagram and description from the previous problem. For each vertex v, let $A(v)$ denote the average value of $d(v)$ for all of the friends of v. What is the average value of $A(v)$ for the seven people, rounded to the nearest tenth?

(A) 2.8 (B) 3.3 (C) 3.5 (D) 3.7 (E) None of these

(13) Suppose 37 students took a test scored between 0 and 100 points, inclusive. The average score was 83. What is the maximum number of students who could have scored under 50?

(A) 36 (B) 12 (C) 18 (D) 6 (E) None of these

(14) Two pieces of broken spaghetti lie on a table. The lengths of the two pieces, in inches, are precisely reciprocals of one another. What is the least possible length, in inches, of the two pieces joined together end to end?

(A) 1 (B) 2 (C) 2.5 (D) 3 (E) None of these

(15) In a double elimination basketball tournament, teams are eliminated once they lose two games. The tournament continues until all but one team has been eliminated, at which point that one remaining team is the champion. If the tournament begins with 43 teams, what is the sum of all possible values of the total number of games played in the tournament?

(A) 42 (B) 43 (C) 84 (D) 169

(E) The answer depends on the design of the tournament.

(16) Venus challenges her sister Serena to a game of ping-pong, the loser of which must buy dinner. Serena knows that she is better than Venus at ping-pong, such that when they play, Serena wins each game independently with probability 70%. However, Serena does not like risk, so to increase her chance of winning, she has the idea of proposing a "best-of-n" series for some odd number n, meaning the two sisters play n games and whoever wins the most gets the free dinner. What is the minimum value of n that gives Serena at least an 85% chance of winning the series?

(A) 3 (B) 5 (C) 7 (D) 9 (E) None of these

(17) A *k-term arithmetic progression* (kAP for short) with common difference d is a set of numbers the form

$$\{x, x + d, \ldots, x + (k - 1)d\}.$$

For example, $\{3, 10, 17, 24\}$ is a 4AP with common difference 7. What is the sum of the terms of the 5AP consisting entirely of prime numbers with the smallest possible common difference?

(A) 35 (B) 85 (C) 335 (D) 28

(E) There are multiple 5APs of primes with the same minimal common difference

(18) Use the definition of kAP from the previous problem. What is the least positive integer n such that every set containing more than half of the integers between 1 and n, inclusive, contains a 3AP?

(A) 6 (B) 8 (C) 10 (D) 12 (E) None of these

(19) A *complex number* takes the form $z = x + iy$, where x (called the *real part* of z) and y (called the *imaginary part* of z) are real numbers, and i is a constant satisfying $i^2 = -1$. What is the largest possible imaginary part of a complex number z satisfying $z^6 = 1$?

(A) 0 (B) $1/2$ (C) $1/\sqrt{2}$ (D) 1 (E) None of these

(20) Using the same definitions as the previous problem, how many complex numbers satisfy $z^{30} = 1$ but $z^n \neq 1$ for all integers $1 \leq n \leq 29$?

(A) 8 (B) 10 (C) 15 (D) 29 (E) None of these

Written Test Solutions

(1) $12^3 = 1728$, while $13^3 = 2197$, the latter of which is closer to 2021. Further, cube roots get closer together as numbers increase, so the answer is 13 **(B)**.

(2) The pairs of integers (a, b) with $a < b$ and $ab = 54$ are $(1, 54)$, $(2, 27)$, $(3, 18)$, $(6, 9)$, the last of which is the only one that satisfies the given condition. Since Austin is the older brother, he is 9 years old **(D)**.

(3) $2021 = 43 \cdot 47$, so the answer is $43 + 47 = 90$ **(A)**.

(4) If a three-dimensional object is scaled by a factor of λ, the object's volume (and hence also its mass if the density is fixed) is multiplied by λ^3. In this case $\lambda = 7$, so the answer is $15 \cdot 7^3 = 5145$ **(D)**.

(5) The units digit of powers of 7 go in a cycle of length four: $7, 9, 3, 1$. Therefore, all that matters is the remainder of the exponent $7^{7^{7^7}}$ when divided by 4. Further, the remainder of powers of 7 when divided by 4 go in a cycle of length two: $3, 1$. Therefore, all that matters is whether the next exponent 7^{7^7} is even or odd. Since this second exponent is odd, the first exponent has a remainder of 3 when divided by 4, so the initial number has units digit 3 **(C)**.

(6) Moving everything to one side we get $x^7 - 8x^4 + 5x^3 - 40 = 0$, which factors as $(x^3 - 8)(x^4 + 5) = 0$. The second factor is never 0 for real x, and the only solution for the second factor is $x = 2$, so the sum is 2 **(E)**.

(7) By the change of base formula, $\log_b a = \frac{\ln a}{\ln b}$, where ln is the natural logarithm (or any other fixed logarithm). Therefore, the given expression is

$$\frac{\ln(3)}{\ln(2)} \cdot \frac{\ln(4)}{\ln(3)} \cdots \frac{\ln(2021)}{\ln(2020)} = \frac{\ln(2021)}{\ln(2)} = \log_2(2021) \ \textbf{(B)}.$$

(8) This problem is simplified by the fact that one of the sides is parallel to the x-axis, so we can think of AC, which has length 8, as the base, and the height is $13 - 1 = 12$. Therefore, the area is $(8)(12)/2 = 48$ **(B)**.

(9) This version does not have the same simplification as the previous. There are many valid approaches, including using the Pythagorean theorem to find the lengths of all three sides (13, $\sqrt{845}$, $\sqrt{1352}$) and applying some version of Heron's formula. However, the quickest approach is to translate the triangle so A is the origin, making $B = (12, 5)$ and $C = (14, 34)$. Then, we can rotate B, which is distance 13 from the origin, to the point $(13, 0)$. Where does this rotation take C? Since $(14, 34) = 2(12, 5) + 2(-5, 12)$, and $(-5, 12)$ gets rotated to $(0, 13)$, we see that C winds up at $(26, 26)$. Therefore, the area of the original triangle is the same as the area of the triangle formed by $(0, 0)$, $(13, 0)$, and $(26, 26)$, which is $13 \cdot 26/2 = 169$ **(A)**.

(10) Looking at the exponent, $x^2+6x+11=(x+3)^2+2$, so the range for this quadratic is $[2,\infty)$, hence the range of the exponent is $(-\infty,-2]$. Using that e^t decreases to 0 as $t\to-\infty$, this implies that the range of $g(x)$ is $(0,e^{-2}]=(0,1/e^2]$ **(A)**.

(11) The seven values of $d(v)$ are: $1,5,4,3,3,2,2$, which add to 20. For an even quicker approach, each edge contributes exactly 2 to the sum of $d(v)$, one for each endpoint, so the sum of $d(v)$ is always twice the number of edges, in this case $2\cdot 10=20$. Therefore, the average is $20/7\approx 2.9$ **(D)**.

(12) The seven values of $d(v)$ are: $5,(1+4+3+2+2)/5=2.4,(5+3+3+2)/4=3.25,(4+5+3)/3=4,(3+4+2)/3=3,(5+4)/2=4.5,(5+3)/2=4$. So the average is $(5+2.4+3.25+4+3+4.5+4)/7\approx 3.7$ **(D)**. The fact that the answer to this problem is noticeably higher than the previous is an example of the *Friendship paradox*, which roughly speaking says that, on average, people's friends have more friends than they do.

(13) Suppose n students score under 50. Then, the sum of the scores is less than $50n+100(37-n)=3700-50n$. Since the average is 83, we know that the sum is exactly $37\cdot 83=3071$. Combining these facts we have $50n<629$, and since n is an integer we have $n\le 12$ **(B)**.

(14) The problem is equivalent to minimizing $x+1/x$ for $x>0$. This can be achieved a number of ways, including with calculus, but a slick algebra observation gives $x+1/x-2=(\sqrt{x}-1/\sqrt{x})^2\ge 0$, with equality if and only if $\sqrt{x}=1/\sqrt{x}$, in other words $x=1$. Therefore, the smallest $x+1/x$ can be is 2 **(B)**.

(15) Every game has one loser. 42 of the 43 teams must lost exactly twice. The champion can either finish with 1 loss or 0 losses. Therefore, the total number of games is either 84 or 85, independent of any other details of the structure of the tournament! So, the sum of all possibilities is 169 **(D)**. This was a modification of the problem of single-elimination tournaments, in which the total number of games is always one less than the number of teams, independent of any other factors.

(16) We repeatedly use that if an event has probability p, the probability of exactly k successes out of n independent trials is $\binom{n}{k}p^k(1-p)^{n-k}$. With $n=3$, the chance Serena loses is $3(.7)(.3)^2+(.3)^3=0.216$. With $n=5$, that chance is $10(.7)^2(.3)^3+5(.7)(.3)^4+(.3)^5=0.16308$. At this point it is a safe prediction that $n=7$ will dip Serena's loss probability below 15%, but to be safe, it is: $35(.7)^3(.3)^4+21(.7)^2(.3)^5+7(.7)(.3)^6+(.3)^7=0.126036$. Therefore, the answer is $n=7$ **(C)**.

(17) If the common difference of a 5AP is odd, then the terms alternate between even and odd numbers, so there are at least two even numbers, at least one of which is composite. If the common difference is not divisible by 3, then the terms cycle through the three possible remainders when divided by 3, so there is either two multiples of

three, at least one of which is composite, or the third of the five terms is a multiple of three. A multiple of 3 can only be prime if it is 3 itself, and 3 can only be the third term of a 5AP of positive integers if the progression is specifically $\{1, 2, 3, 4, 5\}$, which does not qualify since 4 is composite. Therefore, the common difference of a 5AP of primes must be divisible by both 2 and 3, hence it must be divisible by 6. With the example $\{5, 11, 17, 23, 29\}$, we see that 6 is possible. Further, if the progression starts with anything other than 5, and the common difference is not divisible by 5, then by similar logic as before the progression will contain a composite multiple of 5. In summary, a 5AP of primes is either $\{5, 11, 17, 23, 29\}$, or it has common difference greater than 6, so the final answer is $5 + 11 + 17 + 23 + 29 = 85$ **(B)**.

(18) The set $\{1, 2, 4, 5\}$, or truncations thereof, immediately rule out $n \leq 7$. Now consider $n = 8$. If we choose five integers between 1 and 8, inclusive, we must either choose at least three of the first four, or at least three of the last four. By symmetry, let's assume we choose at least three of the first four. The only two choices without forming a 3AP are $\{1, 2, 4\}$ and $\{1, 3, 4\}$. To get up to five total integers, we must now choose at least two of the last four, but all six choices of two of the last four integers form at least one 3AP with each of $\{1, 2, 4\}$ and $\{1, 3, 4\}$. Therefore, it is impossible to choose five numbers in $\{1, 2, 3, 4, 5, 6, 7, 8\}$ without forming a 3AP, so the answer is 8 **(B)**.

(19) Using Euler's formula $e^{it} = \cos(t)+i\sin(t)$, we see that the 6-th roots of 1 are precisely the numbers $1, \omega, \omega^2, \ldots, \omega^5$, where $\omega = e^{i2\pi/6} = e^{i\pi/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. Of this list, ω and ω^2 have the largest imaginary part of $\sqrt{3}/2$, which is best seen by plotting the roots around the unit circle. Therefore, the answer is $\sqrt{3}/2$ **(E)**.

(20) Similar to the previous problem, the 30-th roots of 1 are $1, \omega, \omega^2, \ldots, \omega^{29}$, where $\omega = e^{i2\pi/30} = e^{i\pi/15}$. However, if $z = \omega^k$, and k shares a common factor $d > 1$ with 30, than $z^{30/d} = 1$. In other words, the complex numbers that meet our conditions (called *primitive roots of unity*) take the form $z = \omega^k$ where $\gcd(k, 30) = 1$. Therefore, the answer to the problem is precisely the number of integers $1 \leq k \leq 30$ with $\gcd(k, 30) = 1$ (also known as $\phi(30)$). This can be computed by brute force, with a known formula, or by thinking probabilistically: for $1 \leq k \leq 30$ to be relatively prime to 30, it must satisfy the independent conditions that it is NOT divisible by 2, NOT divisible by 3, and NOT divisible by 5, so the probability is $(1/2)(2/3)(4/5) = 8/30$, and the final answer is 8 **(A)**.