2022 Millsaps College
High School Mathematics Competition

Ciphering Round **Solutions**
10 Problems/3 Minutes Each

- All problems are free response, and 10 points are awarded for each correct answer
- The only things allowed out during the round are the pages from this packet, writing utensils, and scratch paper. In particular, **no calculators or electronic devices of any kind are allowed out during the round.**
- Problems will be worked, and then collected, one at a time. **Do not look at the next page in the packet until directed by a proctor to do so.**
- All work during this round must be done as an individual. **No conversation is allowed during this round.**

Ciphering Round Problems

(1) At Ridgemont High School, 1437 students enjoy playing basketball, while 1198 students enjoy playing tennis. Of those, 613 students enjoy playing both basketball and tennis. How many students at Ridgemont High School enjoy playing at least one of the two sports?

(2) Robbie sold merchandise after his band's latest show. He sold t-shirts for \$18 apiece and hoodies for \$29 apiece, cash only. He also gave each customer a free band sticker for every item they bought. Before he began selling, he had 100 stickers and \$37 in cash. After selling, he had 78 stickers and \$565 in cash. How many hoodies did Robbie sell?

(3) A small pizza has a 12 inch diameter, while a medium pizza has a 16 inch diameter. How many small pizzas total the same area as 18 medium pizzas?

(4) Find the sum of all real solutions to the equation

$$x = 5\sqrt{x} + 14.$$

(5) What is the least positive integer n such that $n! > 4^n$?

(6) Mae and Tommy play a game in which they alternate turns flipping a fair coin, and the first person to flip heads wins. Mae flips first. What is the probability that Mae wins the game? Express your answer as a reduced fraction.

(7) Recall that $\sin^{-1}(x)$ is the unique angle $-\pi/2 \leq \theta \leq \pi/2$ such that $\sin(\theta) = x$, defined whenever such an angle exists. What is the domain of the function

$$f(x) = \sin^{-1}(\sqrt{x} - 2)?$$

Express your answer in interval notation.

(8) The height of a trapezoid is 12cm, and the length of its two diagonals are 13cm and 15cm, respectively. Find the area of the trapezoid, in cm^2.

(9) Find the product of all solutions to the equation

$$\cos(x) = \sin(2x)$$

with $-\pi \leq x \leq \pi$. Provide an exact answer in terms of π.

(10) At Hawkins High School, 953 students enjoy playing basketball, 774 students enjoy playing tennis, and 612 students enjoy playing soccer. Of those, 315 enjoy both basketball and tennis, 246 enjoy both tennis and soccer, and 168 enjoy both basketball and soccer. Finally, 99 students enjoy playing all three sports. How many students at Hawkins High School enjoy playing at least one of the three sports?

Ciphering Round Solutions

(1) Adding the two groups together yields $1437 + 1198 = 2635$ students. However, the 613 students in the intersection have been counted twice, so the correct total is $2635 - 613 = \mathbf{2022}$.

(2) The sticker info tells us that Robbie sold $100 - 78 = 22$ total items, and the cash info tells us that he totaled $565 - 37 = 528$ dollars in revenue. This yields the system of equations $T + H = 22$ and $18T + 29H = 528$, where T and H are the number of t-shirts and hoodies sold, respectively. Substituting $T = 22 - H$ into the second equation yields $11H + 396 = 528$, so $H = \mathbf{12}$.

(3) The radii are in a $3 : 4$ ratio, so the areas are in a $9 : 16$ ratio. In particular, $2 \cdot 9 = 18$ mediums is the same area as $2 \cdot 16 = \mathbf{32}$ smalls.

(4) Moving to one side we have $x - 5\sqrt{x} - 14 = (\sqrt{x} - 7)(\sqrt{x} + 2) = 0$, so $\sqrt{x} = 7$ or $\sqrt{x} = -2$. The latter has no solution, while the former has the single solution $x = \mathbf{49}$.

(5) Carefully multiplying, we see that $8! = 40320 < 65536 = 4^8$, but $9! = 362880 > 262144 = 4^9$, so the answer is $\mathbf{9}$.

(6) Let p denote Mae's probability of winning. Mae wins on the first flip with probability $1/2$. Further, if the first two flips of the game are both tails, which occurs with probability $1/4$, then the game reverts back to its original position. Therefore, p satisfies the equation $p = \frac{1}{2} + \frac{1}{4}p$, so $p = \mathbf{2/3}$.

(7) The range of sine is $[-1, 1]$, so for $f(x)$ to be defined, we must have $-1 \leq \sqrt{x} - 2 \leq 1$, so $1 \leq \sqrt{x} \leq 3$, and hence $1 \leq x \leq 9$. In interval notation, the final answer is $[\mathbf{1}, \mathbf{9}]$.

(8) Drop altitudes from the smaller base to the larger base. This breaks the larger base into a sum of three lengths, a, b, c, ordered left to right, with b the length of the smaller base. In particular, the area is $12(a + b + c + b)/2 = 6(a + 2b + c)$. By the Pythagorean theorem, $(a + b)^2 + 12^2 = 13$, while $(b + c)^2 + 12^2 = 15$, hence $a + b = 5$ and $b + c = 9$. Therefore, $a + 2b + c = 14$, so the area is $\mathbf{84}$ cm^2.

(9) Using the double angle formula $\sin(2x) = 2\sin(x)\cos(x)$ and moving to one side, we have $2\sin(x)\cos(x) - \cos(x) = \cos(x)(2\sin(x) - 1) = 0$, so either $\cos(x) = 0$ or $\sin(x) = 1/2$. In the given range, the former occurs when $x = \pm\pi/2$, while the latter occurs when $x = \pi/6,\ 5\pi/6$. Therefore, the final answer is

$$(\pi/2)(-\pi/2)(\pi/6)(5\pi/6) = \boldsymbol{-5\pi^4/144}.$$

(10) Adding the three sports together yields $953+774+612 = 2339$ students. However, the students in the intersections have been counted more than once, so we must subtract $315+246+168 = 729$. Finally, we note that the 99 students who enjoy all three sports were initially counted three times, but have now been recmoved three times, so we need to add them back in. Therefore, the final answer is $2339-729+99 = \mathbf{1709}$. This problem, and Problem (1), are small cases of the more general phenomenon known as the *inclusion-exclusion principle*.