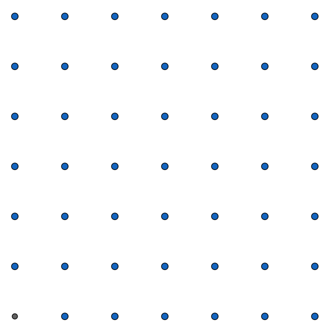
2022 Millsaps College High School Mathematics Competition

Team Round **Solutions**
5 Problems/60 Minutes

- All problems are free response, and 40 points are awarded for each correct answer.
- The only things allowed out during the round are the pages from this packet, writing utensils, and scratch paper. In particular, **no calculators or electronic devices of any kind are allowed out during the round.**
- This round is collaborative. Teammates are encouraged to work together and communicate during the round, but there should not be any communication between teams.

Team Round Problems

(1) How many distinct distances occur between points in a 7×7 square grid, as shown below? To clarify, there are certainly pairs of points that are one unit apart, as well as two units apart, etc, but there are also pairs that are $\sqrt{2}$ apart (one spot up and one spot over). So how many *different* distances are there? For the purposes of this problem, do not include the distance 0 from a point to itself.



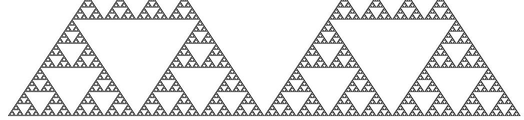
(2) A rectangle R with integer side lengths has perimeter P units and area $2P$ units squared. Find the sum of all possible values of P.

(3) Find the sum of all positive integers n for which $\sqrt{30+\sqrt{n}}+\sqrt{30-\sqrt{n}}$ is an integer.

(4) Suppose six fair, six-sided dice are rolled. What is the probability of rolling exactly three distinct numbers? For example, rolling $(1, 5, 2, 2, 5, 2)$ would count as a success because of the three distinct numbers 1, 5, and 2, while rolling $(5, 5, 2, 2, 5, 2)$ (two distinct numbers) or $(1, 5, 2, 2, 5, 3)$ (four distinct numbers) would not. Express your answer as a reduced fraction.

(5) There are many different ways to conceptualize *dimension* in mathematics. For certain "self-similar" objects, dimension can be thought of in the following way: if one can scale an object by a factor of n, where n is a positive integer, and the result yields n^d copies of my original object, then the object has dimension d. For example, if you scale a cube by a factor of 2, you can chop it up into $8 = 2^3$ copies of the original cube, so the cube has dimension 3.

Below is a picture of the *Sierpiński triangle*, obtained from the following iterative process: start with an equilateral triangle, divide into 4 equilateral triangles with half the original side length, remove the center one, then repeat this process with each of the three remaining triangles as if they were the original big triangle, and on, and on, and on, forever. Using the notion of dimension introduced above, what is the dimension of the Sierpiński triangle? Express your answer as a decimal rounded to the nearest tenth.



Team Round Solutions

(1) By the Pythagorean theorem, all of the distances take the form $\sqrt{a^2+b^2}$ for integers $0 \le a, b \le 6$. By symmetry, we can assume $a \ge b$, and as directed in the problem we exclude $a = b = 0$. Considering $1 \le a \le 6$, this leaves $2+3+4+5+6+7 = 27$ pairs (a, b) for which to compute $a^2 + b^2$, and they all turn out to be distinct except for $5^2 + 0^2 = 4^2 + 3^2$, so the total number of distinct distances is **26**. Specifically, the 26 distances are $\sqrt{n}$ for

$$n = 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, 26, 29, 32, 34, 36, 37, 40, 41, 45, 50, 52, 61, 72.$$

(2) Let x and y, with $x \ge y$, be the integer side lengths of R, so

$$xy = 2P = 2(2x + 2y) = 4x + 4y.$$

This rearranges to $xy - 4x - 4y = 0$. Adding 16 to both sides and factoring we have $(x - 4)(y - 4) = 16$. This leaves three possibilities:

(i) $x - 4 = 16, \;\; y - 4 = 1 \implies P = 2(20) + 2(5) = 50$

(ii) $x - 4 = 8, \;\; y - 4 = 2 \implies P = 2(12) + 2(6) = 36$

(iii) $x - 4 = y - 4 = 4 \implies P = 2(8) + 2(8) = 32.$

Therefore, the final answer is $50 + 36 + 32 =$ **118**.

(3) If the given expression is an integer, then squaring gives

$$\left(\sqrt{30+\sqrt{n}}+\sqrt{30-\sqrt{n}}\right)^2 = 60 + 2\sqrt{900-n} = m^2$$

for some integer m. In other words, $0 \le \sqrt{900-n} < 30$ must be an integer of the form $(m^2 - 60)/2$, so in particular m must be even and m^2 must be between 60 and 120. The only satisfactory choices are $m = 8$ and $m = 10$, which yield $900 - n = 2^2$ and $900 - n = 20^2$, hence $n = 896$ or $n = 500$. Therefore, the final answer is $896 + 500 =$ **1396**.

(4) There are 6^6 possible rolls of six fair, six-sided dice, and we need to count how many result in exactly three distinct numbers. First, we can choose the three numbers, for which we have $\binom{6}{3} = 20$ choices. Once those are chosen, there are three possibilities to consider:

(i) **Two rolls of each number:** In this case we have $\binom{6}{2} = 15$ choices for the two dice to roll the first number, then $\binom{4}{2} = 6$ choices for the second number, totaling $15 \cdot 6 = 90$ possibilities.

(ii) **Three rolls of one number, two rolls of another number, one roll of a third number:** There are $3! = 6$ ways to assign how many rolls there are of each of the three numbers. Then there are $\binom{6}{3} = 20$ options for the dice to roll the first number and $\binom{3}{2} = 3$ options for the second number, totaling $6 \cdot 20 \cdot 3 = 360$ possibilities.

(iii) **Four rolls of one number, one roll apiece for the other numbers:** There are 3 choices for which number gets the four rolls, then $\binom{6}{4} = 15$ options for which dice roll that number, then 2 choices for how to arrange the last two numbers, totaling $3 \cdot 15 \cdot 2 = 90$ possibilities.

Finally, we have $20(90 + 360 + 90) = 20 \cdot 540$ successes, so the final answer is the simplified form of $(20)(540)/6^6$, which is **25/108**.

(5) The key is that if you scale the Sierpiński triangle by 2, the three corner triangles of the scaled version are exact copies of the original triangle, and the middle triangle is removed, so the scaled triangle is precisely comprised of three copies of the original. By the described notion of dimension, this means that the dimension d of the Sierpiński triangle is defined by the equation $2^d = 3$. An exact expression for this dimension would be $\log_2(3)$ or $\ln(3)/\ln(2)$, but this problem has the extra wrinkle that we have to estimate this logarithm to the nearest tenth.

One approach to this estimation is to find powers of 3 and powers of 2 that are close together. Specifically, $3^5 = 2^{5d} = 243$, which is slightly less than $2^8 = 256$, so $5d$ is slightly less than 8, which means d is slightly less than 1.6. If one were concerned whether d is in fact closer to 1.5, we could compare $3^4 = 2^{4d} = 81$, which is more noticeably greater than $2^6 = 64$. This is just one approach that yields the correct final approximation of **1.6**.