2022 Millsaps College High School Mathematics Competition

Written Test **Solutions**
20 Problems/90 Minutes

- All problems are multiple choice with five answer choices. 10 points are awarded for each correct answer, 0 points are awarded for each incorrect answer, and 2 points are awarded for each blank answer.

- The only things allowed out during the round are the test packet, writing utensils, and scratch paper. In particular, **no calculators or electronic devices of any kind are allowed out during the round.**

- All work during this round must be done as an individual. **No conversation is allowed during this round.**

Written Test Problems

(1) What is the integer closest to $\sqrt[4]{2022}$?

(A) 12 (B) 13 (C) 6 (D) 7 (E) None of these

(2) Find the sum of the numerator and denominator of the reduced fraction equal to

$$\frac{0^1 + 1^2 + 2^3 + 3^4}{4^3 + 3^2 + 2^1 + 1^0}.$$

(A) 83 (B) 166 (C) 11 (D) 33 (E) None of these

(3) Sabrina the teenage witch has two aunts, Hilda and Zelda. The two aunts' ages, which are consecutive perfect squares (like 20^2 and 21^2, but not those), add to five times Sabrina's age. Assuming Sabrina really is a teenager, how old is she?

(A) 15 (B) 16 (C) 17 (D) 18 (E) None of these

(4) An ice cream cone has the shape of a perfect right circular cone with base radius 1 inch and height 4 inches. The cone can be completely filled with ice cream, and can also hold an additional hemisphere, also with radius 1 inch, heaping over the top. What volume of ice cream, in cubic inches, can the cone hold in total?

(A) $2\pi/3$ (B) $4\pi/3$ (C) π (D) 2π (E) None of these

(5) The 25 students in Ms. Olson's class have an average of 1.8 siblings. The unique student with the most siblings has 7, while all others have 6 or fewer. What is the maximum possible number of "only children" (students with no siblings) in the class?

(A) 8 (B) 13 (C) 17 (D) 18 (E) None of these

(6) Each car in a fleet of 50 vehicles drives a regular delivery route of 240 miles per week, and each car gets 20 miles per gallon of gas (mpg). To lower costs, the owner replaces half of the cars in the fleet with more efficient models that get 30 mpg. What is the average mpg for the fleet after the old cars are replaced?

(A) 24 (B) 48 (C) 25 (D) 50 (E) None of these

(7) Find the number of trailing zeros in the base-9 expansion of 31!.

(A) 14 (B) 11 (C) 9 (D) 6 (E) None of these

(8) Find the product of all real solutions to the equation

$$\ln\left((x^2-8)^{x+5}\right)\left(e^{3x-6}-1\right)=0.$$

(A) 45 (B) 90 (C) 0 (D) -90 (E) None of these

(9) Find the sum of all real solutions to the equation

$$\cos^7(x)+5\cos^2(x)=10+4\cos(x)$$

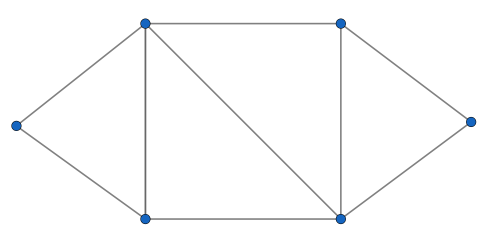
with $-\pi \leq x \leq \pi$.

(A) $\pi/2$ (B) π (C) 0 (at least one solution) (D) 0 (no solutions) (E) None of these

(10) Recall that $\tan^{-1}(x)$ is the unique angle $-\pi/2 < \theta < \pi/2$ satisfying $\tan\theta = x$. What is the range of the function $f(x)=\tan^{-1}(\sqrt{x}-1)$ for $x \geq 0$?

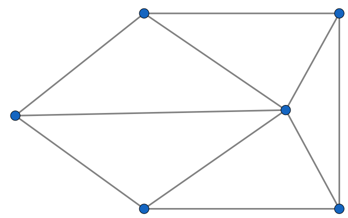
(A) $(-\pi/4, \pi/2)$ (B) $[-\pi/4, \pi/2)$ (C) $[-\pi/4, \pi/4)$ (D) $[-\pi/4, \pi/4]$ (E) None of these

(11) A *graph* is a collection of vertices, some pairs of which are connected by edges. The *chromatic number* of a graph is the minimum number of different colors needed to color the vertices such that no two vertices connected by an edge are the same color. For example, a square, with vertices at the corners, has chromatic number 2, because you could color one diagonal pair of corners red, and the other pair blue. What is the chromatic number of the graph below?



(A) 2 (B) 3 (C) 4 (D) 5 (E) None of these

(12) Using the definition of *chromatic number* introduced in the previous problem, what is the chromatic number of the graph shown below?



(A) 2 (B) 3 (C) 4 (D) 5 (E) None of these

(13) What is the mean of the positive divisors (or factors) of 2022?

(A) 507 (B) 2034/7 (C) 4055/7 (D) 338 (E) None of these

(14) If an integer n is randomly selected between 1 and 1000, inclusive, what is the probability that n is neither a perfect square ($n = m^2$ for an integer m) nor a perfect cube ($n = m^3$ for an integer m)?

(A) 959/1000 (B) 481/500 (C) 24/25 (D) 241/250 (E) None of these

(15) If three fair six-sided dice are rolled, what is the probability of rolling exactly two distinct numbers? For example, $(1, 4, 1)$ would count as a success, but $(1, 1, 1)$ or $(1, 4, 2)$ would not.

(A) 1/36 (B) 5/8 (C) 5/12 (D) 1/6 (E) None of these

(16) A fair six-sided die is rolled, and then a fair coin is tossed a number of times that matches the numbered rolled. For example, if a 3 is rolled, then the coin is flipped three times. If all we know is that this process resulted in no heads being flipped, what is the probability that a 3 was rolled?

(A) 1/8 (B) 1/48 (C) 4/31 (D) 8/63 (E) None of these

(17) Suppose $c \neq 0$ is a real number and the roots of the polynomial $x^2 - cx + 2c$ are $\log_3(k)$ and $\log_5(k)$ for some $k > 0$. Find k.

(A) 15 (B) 1 (C) 1/15 (D) 1/225 (E) None of these

(18) Suppose $k \neq 0$ is a real number and let $P(x) = x^3 + kx + k$. If one root of P is twice another root of P, find the greatest root of P.

(A) 7/2 (B) 16/3 (C) 8 (D) 4 (E) None of these

(19) Suppose a and b are integers, $a \geq b \geq 0$, and $a^2 - b^2 = 1000$. Find the sum of all possible values of a.

(A) 90 (B) 217 (C) 433 (D) 468 (E) None of these

(20) Find the value of e for which the system $a^2 + b^2 + c^2 = d$ and $a + b + c + d = e$ has a unique real solution (a, b, c, d). (For every other value of e, the system either has infinitely many solutions or no solution.)

(A) 3/4 (B) $-3/4$ (C) 9/4 (D) $-9/4$ (E) None of these

Written Test Solutions

(1) Multiplying carefully, we see that $6^4 = 1296$, while $7^4 = 2401$, the latter of which is closer to 2022 . Further, fourth roots get closer together as numbers increase, so the answer is 7 **(D)**.

(2) As written, the numerator is $0+1+8+81 = 90$ and the denominator is $64+9+2+1 = 76$. The fraction $90/76$ reduces to $45/38$, the so the answer is $45 + 38 = 83$ **(A)**.

(3) Since Sabrina's age is between 13 and 19, inclusive, the sum of the aunts' ages is between 65 and 95. The only two consecutive squares whose sum lies in that range are $6^2 = 36$ and $7^2 = 49$. Therefore, Sabrina's age is $(36 + 49)/5 = 17$ **(C)**.

(4) The volume of the cone portion is $\frac{1}{3}\pi(1)^2(4) = 4\pi/3$, while the volume of the hemisphere on top is $\frac{2}{3}\pi(1)^3 = 2\pi/3$, so the total volume is $(4\pi + 2\pi)/3 = 2\pi$ **(D)**.

(5) The total number of siblings amongst the students is $25 \cdot 1.8 = 45$. In order to maximize the number of "only children", we must concentrate those 45 siblings into as few students as possible. One student has 7 siblings, leaving 38. The rest of the students have at most 6 siblings, so we need at least 7 more students to account for the remaining 38 siblings, as $6 \cdot 6 = 36$ is not enough. Therefore, at least 8 students have siblings, leaving at most 17 "only children" **(C)**.

(6) A new car requires $1/30$ of a gallon of gas to drive a mile, while an old car requires $1/20$ of a gallon of gas to drive a mile. Together, they drive two miles using $1/30 + 1/20 = 5/60 = 1/12$ of a gallon of gas, or $1/24$ of a gallon per mile. Therefore, since the new cars and old cars in the fleet drive the same number of miles, and there are the same number of each, the fleet now averages 24 mpg **(A)**.

(7) The question is equivalent to the largest power of 9 that divides $31! = (31)(30)\cdots(2)(1)$. To determine this, since $9 = 3^2$, we need to count the number of 3's in the prime factorization. Every multiple of 3 from 1 to 30, of which there are ten, contributes one factor of 3. Every multiple of 9, of which there are three, contributes an additional factor of 3, and every multiple of 27, of which there is one, contributes an additional factor of 3. Therefore, the total number of 3's is $10 + 3 + 1 = 14$. Since $3^{14} = 9^7$, the answer is 7 **(E)**.

(8) Applying the rules of logarithms we have $(x+5)\ln(x^2-8)(e^{3x-6}-1) = 0$. This means that either $x+5 = 0$, $x^2-8 = 1$, or $3x-6 = 0$, and further, there is a domain restriction that $x^2 - 8 > 0$, as otherwise the logarithm is undefined. The first equation yields $x = -5$, the second equation yields $x = \pm 3$, and the third equation yields $x = 2$, but that final candidate does not meet the domain requirement. Therefore, the only solutions are $x = -5, -3, 3$, so the answer is $(-5)(-3)(3) = 45$ **(A)**.

(9) Since $-1 \leq \cos(x) \leq 1$ for all x, the left hand side is always at most 6, with equality holding only when $\cos(x) = 1$. Meanwhile, the right hand side is always at least 6, with equality holding only when $\cos(x) = -1$. Since 6 is the only point in both ranges, and they cannot both reach it simultaneously, there are no solutions to the equation **(D)**.

(10) We start with $f(0) = \tan^{-1}(-1)$, which is the angle $-\pi/2 < \theta < \pi/2$ with $\tan\theta = -1$, meaning $\sin\theta$ and $\cos\theta$ are opposites, which is $\theta = -\pi/4$. Then, as the x increases, $\tan\theta = \sqrt{x} - 1$ also increases. In order for $\tan\theta$ to continue to grow, $\sin\theta$ must approach 1 while $\cos(\theta)$ approaches 0, so θ approaches, but never achieves $\pi/2$. Therefore, the range is $[-\pi/4, \pi/2)$ **(B)**.

(11) A key observation is that if a graph contains a triangle, then to avoid having connected vertices that are the same color, all three vertices of the triangle must be different colors. Since this graph contains a triangle, we need at least three colors. But, we can do the job with exactly three colors, for example one can color the vertices, around the outside clockwise starting in the bottom left corner of the square, blue, red, green, blue, green, red. Therefore, the chromatic number is 3 **(B)**.

(12) First we observe that three colors is not enough. By the key observation in the previous problem's solution, any triangle in the graph has to be colored with three different colors. In this graph, that will subsequently force your hand on all of other vertices, eventually leading to a dead end. However, four colors is enough. Color the central vertex red, then color the five outer vertices, in either clockwise or counterclockwise order, blue, green, yellow, blue, green. Therefore the chromatic number is 4 **(C)**. It is a famous theorem that every *planar* (can be drawn in a plane without intersecting edges) graph has chromatic number at most 4.

(13) The prime factorization of 2022 is $2 \cdot 3 \cdot 337$ (to verify that 337 is prime, it suffices to check for divisibility by primes up to 17). Therefore, the eight divisors of 2022, in increasing order, are $1, 2, 3, 6, 337, 674, 1011, 2022$. The mean of these numbers is $(1 + 2 + 3 + 6 + 337 + 674 + 1011 + 2022)/8 = 4056/8 = 507$ **(A)**.

(14) The largest perfect square that is at most 1000 is $31^2 = 961$, so there are 31 perfect squares. The largest such perfect cube is exactly $10^3 = 1000$, so there are 10 perfect cubes. A number is both a square and a cube if and only if it is a perfect sixth power, of which there are 3 that are at most 1000. Therefore, the union of the squares and the cubes has $31 + 10 - 3 = 38$ elements, so the probability of avoiding them is $(1000 - 38)/1000 = 962/1000 = 481/500$ **(B)**.

(15) The total number of possibilities when rolling three dice is $6^3 = 216$, so we need to count the successes. To have exactly two distinct numbers, there must be two dice with one number and one die with the other number. There are 6 choices for the number with two dice, and then 5 choices for the number with one die. Once this choice is made, there are 3 choices for the one die to roll the second number, yielding a total of $6 * 5 * 3 = 90$ successes. Therefore, the answer is $90/216 = 5/12$ **(C)**.

(16) If n is rolled, the chance of no heads being flipped is $1/2^n$. Therefore, the total probability of no heads being flipped is $(1/6)(1/2 + 1/4 + 1/8 + 1/32 + 1/64)$. Of this, $(1/6)(1/8)$ comes from a roll of 3, so as a proportion of the total "no heads" probability, the "roll 3 and no heads" probability is

$$(1/8)/(1/2 + 1/4 + 1/8 + 1/32 + 1/64) = (1/8)(64/63) = 8/63 \ \ \textbf{(D)}.$$

(17) Expanding out $(x - \log_3(k))(x - \log_5(k))$, we see that $\log_3(k) + \log_3(k) = c$, while $\log_3(k)\log_5(k) = 2c$. In other words,

$$\log_3(k)\log_5(k) = 2(\log_3(k) + \log_3(k)) \neq 0.$$

We now exploit two properties of logarithms: $\log_b(a) = 1/\log_a(b)$ for $a, b > 0$, and $\log_b(x) + \log_b(y) = \log_b(xy)$ for $b, x, y > 0$. Dividing by the left hand side in the display equation above, we have

$$1 = 2(1/\log_5(k) + 1/\log_3(k)) = 2(\log_k(5) + \log_k(3)) = 2\log_k(15).$$

Finally, we divide by to get $2 = 1/\log_k(15) = \log_{15}(k)$, hence $k = 15^2 = 225$ **(E)**.

(18) The x^2 coefficient of a cubic polynomial is the negative sum of its roots, so in this case the roots of P add to 0. Since one root is double another root, the three roots take the form r, $2r$, and $-3r$, and since $k \neq 0$,, we know $r \neq 0$. Therefore, we have $P(x) = (x - r)(x - 2r)(x + 3r) = x^3 - 7r^2x + 6r^3$. Since the last two coefficients both equal k, we know $-7r^2 = 6r^3$, and since $r \neq 0$, we conclude $r = -7/6$. Finally, the largest root is $-3r = 7/2$ **(A)**.

(19) Since $a^2 - b^2 = (a + b)(a - b)$, the average of the two factors is a, and the difference between those factors is $2b$, we can make this equal 1000 given any factorization of 1000 into a product of two even positive integers. This can be done in four ways: $2 \cdot 500$, $4 \cdot 250$, $10 \cdot 100$, $20 \cdot 50$. This yields the possible values 251, 127, 55 and 35 for a, so the answer is $251 + 127 + 55 + 35 = 468$ **(D)**.

(20) Using the first equation to substitute for d in the second equation, we have

$$a^2 + a + b^2 + b + c^2 + c + d^2 + d = e.$$

Completing the square in all three variables, we have

$$(a + 1/2)^2 + (b + 1/2)^2 + (c + 1/2)^2 = e + 3/4.$$

If the right hand side is negative, this is impossible. If the right hand is positive, this has infinitely many real solutions. If the right hand side is 0, then $a = b = c = -1/2$ is the unique solution. Therefore, the answer is $e = -3/4$ **(B)**.