2023 Millsaps College
High School Mathematics Competition

Team Round **Solutions**
5 Problems/60 Minutes

- All problems are free response, and 40 points are awarded for each correct answer. Problem 1 has four parts, which are worth 10 points apiece.

- The only things allowed out during the round are the pages from this packet, writing utensils, and scratch paper. In particular, **no calculators or electronic devices of any kind are allowed out during the round.**

- This round is collaborative. Teammates are encouraged to work together and communicate during the round, but there should not be any communication between teams.

Team Round Problems

(1) A *simple, closed curve* in the plane is like a circle, except it is allowed to wander around however it wants, in any shape, as long as it does not cross over itself (simple), and finishes where it started (closed). The Jordan Curve Theorem says that any simple, closed curve separates the plane into exactly two regions, one of which is bounded (the *inside*) and one of which is unbounded (the *outside*). This may seem intuitively obvious, but it is nontrivial to prove!

In the simple, closed curve pictured below, determine if each of the labeled points A, B, C, D are in the inside or outside of the curve.

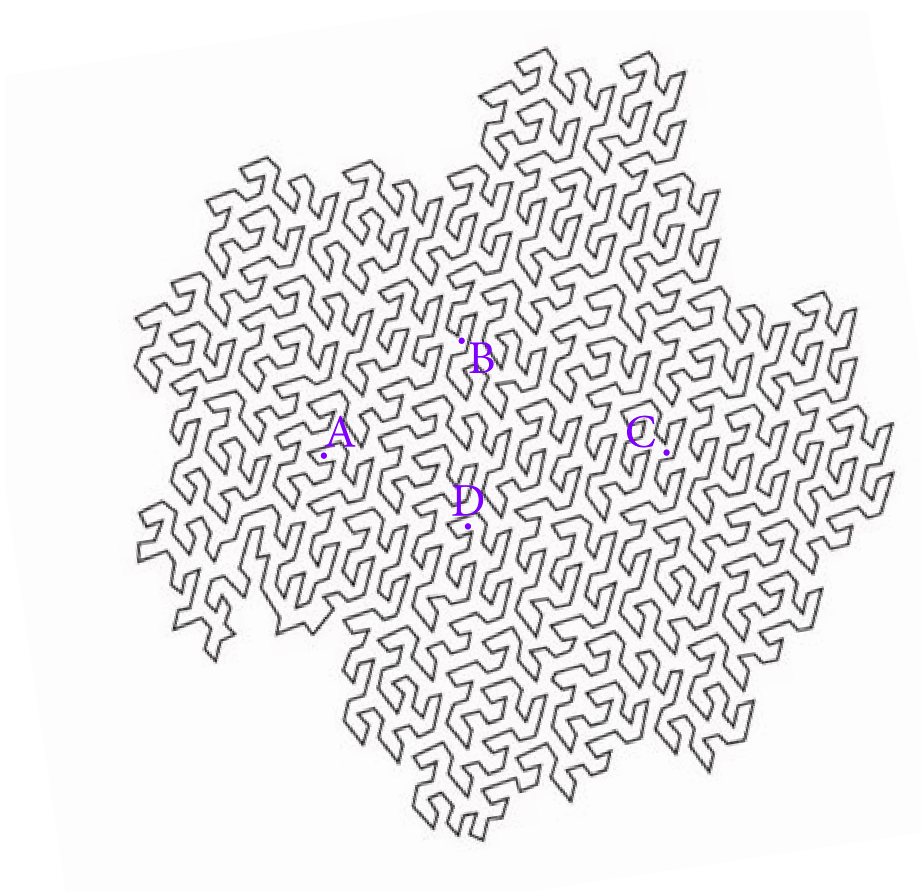

Image Credit: *Topology and Groupoids* by Ronald Brown

(2) In her calculus class, Mackenzie has a 100% homework average, which counts for 20% of her final grade. She has an 95% quiz average, which counts for 20% of her final grade. Her four regular test scores (out of 100), which each count for 12% of her final grade, are 76, 84, 91, and 88. All that remains is the final exam, which is worth 12% of the final grade, her score on which can also replace her lowest regular test score. What does Mackenzie need to score on the final exam (out of 100) to earn a final grade of 90%?

(3) For a set of real numbers A, we define the *sumset* $A+A$ to be the set of all sums of two elements of A, including elements added with themselves. For example, if $A = \{2, 5, 13\}$, then $A + A = \{4, 7, 10, 15, 18, 26\}$. Similarly, the *product set* AA would be $\{4, 10, 25, 26, 65, 169\}$.

If A contains 6 positive real numbers, what is the minimum possible size of the *larger* of the two sets $A+A$ and AA?

For example, if $A+A$ has 20 elements, and AA has 23 elements, then the important number is 23. You want to keep that larger size as *low* as possible. (A rigorous proof is difficult, but determining and convincing yourself of the answer is doable.)

(4) In a regular game of tennis, a player wins the game when they have scored at least four points, and at least two more points than their opponent. Alex and Jamie play a modified game called "Major tennis", in which a player wins the game when they have scored at least three points, and at least three more points than their opponent.

Because Jamie is better than Alex at tennis, Jamie wins each point independently with probability $2/3$. What is the probability that Jamie wins the game of Major tennis? Express your answer as a reduced fraction.

(5) How many ways can the integers 1 thru 20, inclusive, be each colored red or blue, such that an integer is never the same color as its double or its triple?

Team Round Solutions

(1) One can certainly approach and complete this problem like a good, old-fashioned maze. However, a shortcut is to note that if you draw a path from a labeled point, every time you cross over the curve, you switch between the inside and outside. Therefore, if you can escape the picture after an *odd* number of crossings, then the point must have originated on the *inside*, while if you can escape the picture after an *even* number of crossings, the point must have originated on the *outside*.

This technique more quickly uncovers that A **is on the outside, while** B**,** C**, and** D **are all on the inside.**

(2) Mackenzie earned all 20 available points from homework, and $95\% \cdot 20 = 19$ available points from quizzes, so she needs $90 - 20 - 19 = 51$ of the 60 available points from tests. Since $51/60 = 85\%$, and the four regular tests and the final are worth the same, her average on the five tests must be 85, meaning they must add to $85 \cdot 5 = 425$. Her four tests so far total $76 + 84 + 91 + 88 = 339$. Without the wrinkle of the final exam replacing the lowest regular test score, she would need $425 - 339 = 86$ on the final, but instead, she needs twice her final exam score, plus $84 + 91 + 88 = 263$, to equal 425. In other words, she must score half of $425 - 263 = 162$, which is **81**.

(3) Hopefully this problem led to some good trial and error and experimentation! A key is that the sumset is small if there are a lot of *repeated sums*, while the product set is small if there are a lot of *repeated products*. A good way to get a lot of both is to use small positive integers, and to limit the number of distinct prime divisors. Putting these ideas together, a good candidate is $A = \{1, 2, 3, 4, 6, 8\}$, which has

$$A + A = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16\}$$

and

$$AA = \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 32, 36, 48, 64\}.$$

The larger of these is AA, which has **15** elements, and it turns out to be impossible to do any better!

Proof of this fact is among the results of this recent paper, co-authored by five Millsaps students, recently accepted for publication: `https://arxiv.org/abs/2307.06874`

(4) Let p be the probability that Jamie wins given that he is up by one point, and let q be the probability that Jamie wins given that he is down by one point. Since Jamie has a $2/3$ chance of winning the first point, the final answer will be $\frac{2}{3}p + \frac{1}{3}q$, so the task is reduced to determining p and q.

Suppose Jaime is up by one point. If he wins the next two points (probability $4/9$), he wins the game. If he wins one of the next two points (probability $4/9$), he remains up by one. If he loses the next two points (probability $1/9$), he is down by one point. Therefore, $p = \frac{4}{9} + \frac{4}{9}p + \frac{1}{9}q$, which rearranges to $5p - q = 4$.

Suppose Jaime is down by one point. If he wins the next two points (probability $4/9$), he is up by one point. If he wins one of the next two points (probability $4/9$), he remains down by one point. If he loses the next two points, he loses the game. Therefore, $q = \frac{4}{9}p + \frac{4}{9}q$, which rearranges to $5q = 4p$.

We now have a system of two equations for p and q. Adding 5 times the first equation to the second yields $25p = 20 + 4p$, hence $p = 20/21$, and further $q = 16/21$.

Finally, the answer is $\frac{2}{3} \cdot \frac{20}{21} + \frac{1}{3} \cdot \frac{16}{21} = \frac{56}{63} = \mathbf{\frac{8}{9}}$.

(5) Integers not divisble by 2 or 3 can be colored arbitrarily. However, once those integers are colored, the colors of all remaining integers are forced upon you. For example, if 1 is blue, then 2 and 3 must both be red, so 4, 6, and 9 must all be blue, so 8, 12, and 18 must all be red, and finally 16 must be blue. More generally, every integer has a unique factorization of the form $n = 2^a 3^b m$, where m is not divisible by 2 or 3, and the color of n is completely determined by the color of m and whether $a + b$ is even or odd.

Between 1 and 20, inclusive, the numbers not divisible by 2 or 3 are $1, 5, 7, 11, 13, 17, 19$. Since there are 7 such numbers, they can be each colored red or blue in $2^7 = \mathbf{128}$ different ways.