2023 Millsaps College
High School Mathematics Competition

Written Test **Solutions**
20 Problems/90 Minutes

- All problems are multiple choice with five answer choices. 10 points are awarded for each correct answer, 0 points are awarded for each incorrect answer, and 2 points are awarded for each blank answer.
- The only things allowed out during the round are the test packet, writing utensils, and scratch paper. In particular, **no calculators or electronic devices of any kind are allowed out during the round.**
- All work during this round must be done as an individual. **No conversation is allowed during this round.**

Written Test Problems

(1) Zapoli's pizzeria measures its pizzas by diameter. A 10-inch pepperoni pizza sells for \$10. If a 14-inch pepperoni pizza is priced the same per unit area as the 10-inch, how much does the 14-inch pepperoni pizza cost?

(A) \$14 (B) \$19.60 (C) \$18 (D) \$24.40 (E) None of these

(2) What is the largest number of cents for which one can make exact change with only nickels, but *not* with only dimes and quarters?

(A) 5 (B) 15 (C) 55 (D) There are infinitely many such numbers (E) None of these

(3) A sock drawer contains exactly 20 socks, each of which is either blue or red. If an additional five blue socks are added, the probability of randomly drawing a blue sock is 44%. What is the original probability (before the extra blue socks are added) of randomly drawing a red sock?

(A) 30% (B) 56% (C) 60% (D) 70% (E) None of these

(4) The *radical* of a positive integer n is the product of all *distinct* prime divisors of n. For example, the radical of 360 is $2 \cdot 3 \cdot 5 = 30$. What is the radical of 2023?

(A) 2023 (B) 143 (C) 119 (D) 77 (E) None of these

(5) What is the area of a right triangle with one leg length 20 and hypotenuse 29?

(A) 210 (B) 290 (C) 315 (D) 420 (E) None of these

(6) What is the least integer that can be written as a sum of two different positive perfect squares in two different ways (other than swapping order)?

(A) 5 (B) 25 (C) 50 (D) 65 (E) None of these

(7) If $2023^\circ = x$ radians, what is the integer closest to x?

(A) 35 (B) 36 (C) 115909 (D) 115910 (E) None of these

(8) The Chattahoochee High School women's basketball team has 8 players, and the coach must select 5 players for the starting lineup. Two of the players are sisters who are currently in a fight, and refuse to be in the starting lineup *together*. Apart from this restriction, all other starting lineups are permissible (one sister starting is ok, no sisters starting is ok). How many permissible starting lineups are there?

(A) 6 (B) 20 (C) 36 (D) 56 (E) None of these

(9) Mike and Ellen each walk the exact same route through an airport. The route includes some regular ground, then a moving sidewalk (moving toward their destination), then some more regular ground. They each walk the exact same constant speed, and they each pause to tie their shoe for the same amount of time. The only difference is that Mike ties his shoe on the moving sidewalk, while Ellen ties her shoe on regular ground. Who completes the route more quickly?

(A) Mike (B) Ellen

(C) Depends on the speeds of the walking and the moving sidewalk.

(D) Depends on the lengths of the regular ground and the moving sidewalk.

(E) Depends on both the speeds and the lengths.

(10) Cory has written n letters of the alphabet. Three of the letters are S, while the other $n-3$ letters are all distinct and not S. There are over 1000 distinguishable ways to put the letters in order. What is the minimum possible value of n?

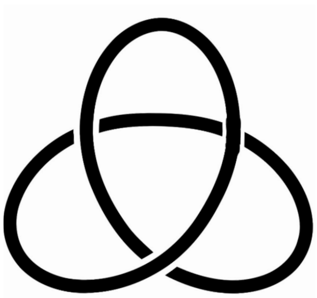
(A) 7 (B) 8 (C) 9 (D) 10 (E) None of these

(11) If a and b are nonzero real solutions to the equation $x^2 + ax + 3b = 0$, what is b?

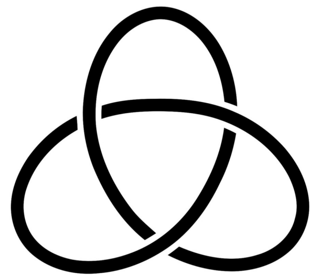
(A) 3 (B) 6 (C) -3 (D) -6 (E) None of these

(12) In mathematical knot theory, a *knot* is a curve in three-dimensional space that does not cross through itself (*simple*) and ends where it begins (*closed*). A *trivial knot* (also called an *unknot*) is a knot that can be "untangled" into a regular circle without "breaking" the curve (think of it as made out of string).

Which of the knots depicted below are trivial?



(I)



(II)

(A) I only (B) II only (C) I and II (D) neither

(13) Consider a strictly increasing list of 23 positive integers $n_1 < n_2 < \cdots < n_{22} < n_{23}$. If the mean, M, of the list satisfies $n_{22} < M < n_{23}$, what is the least possible value of n_{23}?

(A) 23 (B) 24 (C) 252 (D) 253 (E) None of these

(14) What is the product of all solutions to $\tan^2(x) = 3$ for $0 \leq x < 2\pi$?

(A) $\dfrac{40\pi^4}{81}$ (B) $\dfrac{385\pi^4}{1296}$ (C) $\dfrac{105\pi^4}{256}$ (D) $\dfrac{55\pi^4}{81}$ (E) None of these

(15) Back to the Chattahoochee High School women's basketball team. The sisters have now made up, so all starting lineups of 5 players from the 8-player roster are allowed. The players' jersey numbers are $9, 12, 17, 23, 24, 33, 42,$ and 55. If a starting lineup is selected at random, what is the expected value of the *highest* jersey number in the starting lineup?

(A) 33 (B) 47.5 (C) 49 (D) 55 (E) None of these

(16) In the non-right triangle ABC, $\cos(A)\cos(B) = \cos(C)$. Compute the value of $\tan(A)\tan(B)$.

(A) 1 (B) $\sqrt{3}$ (C) 3 (D) 4 (E) None of these

(17) How many subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ contain at least one prime number?

(A) 4 (B) 32 (C) 480 (D) 512 (E) None of these

(18) If x, y, and z are positive real numbers such that

$$\frac{x}{y} + \frac{y}{x} = \frac{17}{z}, \quad \frac{y}{z} + \frac{z}{y} = \frac{13}{x}, \quad \text{and} \quad \frac{z}{x} + \frac{x}{z} = \frac{5}{y},$$

what is the value of $x + y + z$?

(A) 19/2 (B) 10 (C) 11 (D) 23/2 (E) None of these

(19) How many integers between 1 and 2023, inclusive, have a prime divisor between one-half and one-third of $\sqrt{2023}$?

(A) 106 (B) 119 (C) 219 (D) 225 (E) None of these

(20) What is the least nonnegative solution to $\cos(x)\cos(2x)\cos(4x) = 1/8$?

(A) $\pi/9$ (B) $\pi/7$ (C) $2\pi/9$ (D) $2\pi/7$ (E) None of these

Written Test Solutions

(1) Since area is two-dimensional, while diameter is one-dimensional, scaling the diameter by 1.4 scales the area by $1.4^2 = 1.96$, so the 14-inch pizza costs \$19.60 **(B)**

(2) Every multiple of 5 cents is possible with only nickels, while 15 cents is impossible with only dimes and quarters. Further, for any multiple of 5 cents above 15, the amount is either divisible by 10, hence possible with only dimes, or ends in 5 and is at least 25, hence attainable with one quarter and the rest dimes. **(B)**

(3) With all 25 socks, the number of blue socks is 44% of 25, which is 11. Therefore, there were originally $11 - 5 = 6$ socks, hence $20 - 6 = 14$ red socks, so the original probability of drawing a red sock is $14/20 = 70\%$. **(D)**

(4) $2023 = 7 \cdot 17^2$, so the radical is $7 \cdot 17 = 119$ **(C)**

(5) By the Pythagorean theorem, the length of the other leg is $\sqrt{29^2 - 20^2} = 21$, so the area is $(20)(21)/2 = 210$. **(A)**

(6) If 0 were allowed, the answer would be $25 = 0^2 + 5^2 = 3^2 + 4^2$. If repeats were allowed, the answer would be $50 = 5^2 + 5^2 = 1^2 + 7^2$. But, under the given criteria, the smallest is $65 = 1^2 + 8^2 = 4^2 + 7^2$. **(D)**

(7) $2023\pi/180 \approx 35.308$, so the closest integer is 35. **(A)**

(8) There are $\binom{8}{5} = 56$ total possible starting lineups, but $\binom{6}{3} = 20$ of them include both sisters, so only $56 - 20 = 36$ are permissible. **(C)**

(9) Imagine Mike and Ellen are walking side-by-side, except that Ellen stops to tie her shoe a split second before she steps on the moving sidewalk, while Mike waits until a split second after. Mike will gain a lead while the two tie their shoes, which he will never relinquish, regardless of the numerical details of the speeds or lengths. **(A)**

(10) The number of distinguishable ways to put the letters in order is $n!/3! = n!/6$, so we need the smallest n such that $n! > 6000$. Since $7! = 5040$ and $8! = 40320$, the answer is 8. **(B)**

(11) If a and b are the roots, then $x^2 + ax + 3b = (x - a)(x - b) = x^2 - (a + b)x + ab$, so $3b = ab$, hence $a = 3$. Then, $-(3 + b) = 3$, so $b = -6$. **(D)**

(12) Hopefully this was a fun exercise in spatial reasoning! Knot (I) is just a circle that has been twisted in the front, so it can be easily untwisted, while Knot (II) is a *trefoil knot*, which is nontrivial. **(A)**

(13) To keep everything as small as possible, start the list with $1, 2, \ldots, 22$, which add to $23 \cdot 11 = 253$. The mean is then $M = (253 + n_{23})/23$, which needs to be greater than 22. Solving this inequality yields $n_{23} > 253$, so the least possible value is 254. **(E)**

(14) The equation holds if $\tan(x) = \pm\sqrt{3}$. The positive occurs at $x = \pi/3, 4\pi/3$, while the negative occurs at $x = 2\pi/3, 5\pi/3$, so the product of all solutions is $40\pi^4/81$. **(A)**

(15) Of the $\binom{8}{5} = 56$ total starting lineups, $\binom{7}{4} = 35$ of them have #55, $\binom{6}{4} = 15$ of them have #42 as the highest number, $\binom{5}{4} = 5$ of them have #33 as the highest number, and $\binom{4}{4} = 1$ has #24 as the highest number. Therefore, the expected value is $(35 \cdot 55 + 15 \cdot 42 + 5 \cdot 33 + 24)/56 = 49$. **(C)**

(16) Since the angles of a triangle add to π, we have $C = \pi - A - B$. Using properties of cosine and an angle-sum identity, we get

$$\cos(C) = \cos(\pi - A - B) = -\cos(A + B) = -\cos(A)\cos(B) + \sin(A)\sin(B).$$

Therefore, the given relation implies $2\cos(A)\cos(B) = \sin(A)\sin(B)$. Dividing yields $2 = \tan(A)\tan(B)$. **(E)**

(17) There are $2^9 = 512$ total subsets, only $2^5 = 32$ of which are contained in $\{1, 4, 6, 8, 9\}$, so the answer is $512 - 32 = 480$. **(C)**

(18) Multiply the first equation by z, the second by x, and the third by y. This yields

$$xz/y + yz/x = 17, \; xy/z + xz/y = 13, \; yz/x + xy/z = 5.$$

Summing the equations and dividing by 2 yields $xy/z + yz/x + xz/y = 35/2$, and subtracting the three previous equations gives $xy/z = 1/2$, $yz/x = 9/2$, $xz/y = 25/2$. Multiplying these yields $xyz = 225/8$ and then dividing this by the three previous equations yields $z^2 = 225/4$, $x^2 = 25/4$, $y^2 = 9/4$, so $z = 15/2$, $y = 5/2$, $x = 3/2$. Finally, $x + y + z = 23/2$. **(D)**

(19) The square root of 2023 is around 45, so the only primes between one-half and one-third of $\sqrt{2023}$ are 17 and 19. Between 1 and 2023, inclusive, there are $2023/17 = 119$ multiples of 17, $\lfloor 2023/19 \rfloor = 106$ multiples of 19, and $\lfloor 2023/323 \rfloor = 6$ multiples of *both* 17 and 19, so the final answer is $119 + 106 - 6 = 219$. **(C)**

(20) Multiply both sides by $\sin x$ and repeatedly apply the double-angle formula for sine to obtain $\sin x = \sin 8x$. Periodicity properties then imply either $7x = 2k\pi$ or $9x = (2k+1)\pi$ for some integer k. Enumerating the possibilities yields $0, 2\pi/7, 4\pi/7, 6\pi/7$ and $\pi/9, 3\pi/9, 5\pi/9, 7\pi/9, \pi$. We observe that the solutions $x = 0, \pi$ are extraneous since we multiplied by $\sin x$ at the beginning, so the smallest nonnegative true solution is $\pi/9$. **(A)**