2024 Millsaps College
High School Mathematics Competition

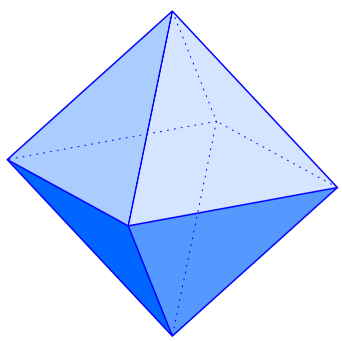
Team Round **Solutions**
5 Problems/60 Minutes

- All problems are free response, and 40 points are awarded for each correct answer. Problem 1 has four parts, which are worth 10 points apiece.
- The only things allowed out during the round are the pages from this packet, writing utensils, and scratch paper. In particular, **no calculators or electronic devices of any kind are allowed out during the round.**
- This round is collaborative. Teammates are encouraged to work together and communicate during the round, but there should not be any communication between teams.

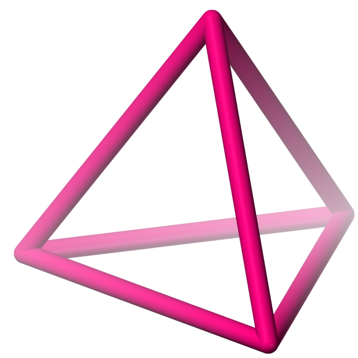
Team Round Problems

(1) What is the maximum number of integers that can be chosen between 1 and 12, inclusive, such that no two chosen integers differ by a prime number?

(2) Let V_1 be the volume of a regular octahedron (eight faces) with edge length 1. Let V_2 be the volume of a regular tetrahedron (four faces) of edge length 2. What is V_2/V_1?



(A) regular octahedron



(B) regular tetrahedron

(3) An integer n has the following properties:

(i) n is divisible by every single digit prime number

(ii) n is divisible by only single digit prime numbers

(iii) The number of positive divisors of n is NOT divisble by 3

(iv) $n \leq 10000$

What is the largest possible value of n?

(4) Simplify the following quantity to the form $a + b\sqrt{d}$, where a and b are reduced fractions, and $d > 1$ is an integer with no perfect square divisors (other than 1):

$$\frac{2-\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}} + \frac{2+\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}$$

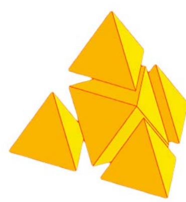
(5) Find the probability that a random permutation of the word MISSISSIPPI contains at least one of IIII (meaning all 4 I's occur consecutively), SSSS, or PP. Express your answer as a percentage, rounded to the nearest percent.

Team Round Solutions

(1) It is possible to choose **4** numbers and satisfy the condition, for example $\{1, 2, 10, 11\}$, but it is impossible to choose any more. You can convince yourself through trial and error that more than 4 is impossible, allowing you to correctly answer the problem, but a rigorous proof requires more work, which we provide below:

Assume without loss of generality that you select 1, as otherwise you can slide all of your selections backward so that 1 is included. Then, 3, 4, 6, 8, and 12 are disallowed, leaving only $2, 5, 7, 9, 10, 11$ available. Further, the only way to select two numbers out of $2, 5, 7, 9$ is to select 5 and 9, in which case both 10 and 11 are disallowed, leaving you with only three selections. If instead you select only one number out of $2, 5, 7, 9$, then the best you can do is to select both 10 and 11, leaving you with a total of four selections, which completes the proof.

(2) The key observation is that one can glue four tetrahedra with edge length 1 onto the octahedron of edge length 1 to form the tetrahedron of edge length 2, as pictured below. Since volume is three dimensional, the volume of each tetrahedron of edge length 1 is $V_2/8$, hence $V_1 + 4(V_2/8) = V_2$, which yields $V_2/V_1 = \mathbf{2}$.



Credit Note: I got the above picture, and the concept for this problem, from the 2013 University of Georgia HS math competition.

(3) If n is divisible by every single digit prime number, then n is divisible by $(2)(3)(5)(7) = 210$, say $n = 210k$ for an integer k. Since n is *only* divisible by single digit prime numbers, the same is true for k, and since n is at most 10000, we know k is at most 10000/210, which rounds down to 47. The largest such values of k, in decreasing order, are 45, 42, 40, and 36. However, the first three listed choices of k yield 5^2, 7^2, and 5^2, respectively, in the prime factorization of n, which makes the number of positive divisors of n divisible by 3. This can be seen by the formula that gives the number of positive divisors of $p_1^{a_1} \cdots p_j^{a_j}$ as $(a_1 + 1) \cdots (a_j + 1)$, or more directly by noting that if p^2 appears in the factorization of n, then there are equal number of divisors of n with p^0, p^1, and p^2, respectively, in their factorizations.

Therefore, the final answer is $n = 210 \cdot 36 = 2^3 \cdot 3^3 \cdot 5 \cdot 7 = \mathbf{7560}$, which has $(4)(4)(2)(2) = 64$ positive divisors.

(4) Let $a = \sqrt{2+\sqrt{3}}$ and $b = \sqrt{2-\sqrt{3}}$. Note that $a+b$ and $a-b$ are both positive, $(a+b)^2 = 6$, and $(a-b)^2 = 2$, so $a+b = \sqrt{6}$ and $a-b = \sqrt{2}$. Further, $ab = 1$, so $a^2+b^2 = (a+b)^2 - 2ab = 4$. Then, adding $b^2/(\sqrt{2}+a)$ to $a^2/(\sqrt{2}-b)$ under the common denominator $2+\sqrt{2}(a-b)-ab = 3$, we have

$$\begin{aligned}\frac{b^2(\sqrt{2}-b)+a^2(\sqrt{2}+a)}{3} &= \frac{\sqrt{2}(a^2+b^2)+a^3-b^3}{3}\\ &= \frac{4\sqrt{2}+(a-b)(a^2+ab+b^2)}{3}\\ &= \frac{4\sqrt{2}+5\sqrt{2}}{3} = \mathbf{3\sqrt{2}}.\end{aligned}$$

Credit Note: A similar problem appeared in a University of Vermont high school math competition in 2020.

(5) There are a few possible approaches here; I will take the approach of treating all 11 letters as distinguishable from one another (imagine the four I's are all different colors, etc.), so the sample space is all 11! possible permutations.

How many of the permutations contain IIII? First treating IIII as a single character, there are now 8 characters to put in order, so 8! permutations, but for each of these the four I's can be reordered 4! ways, for a total of 8!4! permuations containing IIII. Similarly, there are 8!4! permutations containing $SSSS$, and 10!2! permutations containing PP.

If we just add these three counts together, we will double count permutations that have more than one of IIII, SSSS, or PP. More precisely, we need to use the *inclusion-exclusion principle*, which says that the number of permutations with at least one of the three patterns is

$$\begin{aligned}\#(\text{IIII or SSSS or PP}) =& \#\text{IIII} + \#\text{SSSS} + \#\text{PP} - \#(\text{IIII and SSSS}) - \#(\text{IIII and PP})\\ &- \#(\text{SSSS and PP}) + \#(\text{IIII and SSSS and PP}).\end{aligned}$$

Again treating the blocks of consecutive letters as single characters, then multiplying by the number of reorderings, we have #(IIII and SSSS) = 5!4!4!, #(IIII and PP) = #(SSSS and PP) = 7!4!2!, and #(IIII and SSSS and PP) = 4!4!4!2!, for a final probability of

$$p = \frac{8!4! + 8!4! + 10!2! - 5!4!4! - 7!4!2! - 7!4!2! + 4!4!4!2!}{11!}.$$

What remains is to round p to the nearest percent without the aid of a calculator. Cancelling all factors that appear in every term, one can reduce down to

$$p = \frac{280 + 1050 - 10 - 35 - 35 + 4}{5775} = \frac{1254}{5775} \approx 0.217,$$

so the final answer is **22**%.