2024 Millsaps College
High School Mathematics Competition

Written Test **Solutions**
20 Problems/90 Minutes

- All problems are multiple choice with five answer choices. 10 points are awarded for each correct answer, 0 points are awarded for each incorrect answer, and 2 points are awarded for each blank answer.

- The only things allowed out during the round are the test packet, writing utensils, and scratch paper. In particular, **no calculators or electronic devices of any kind are allowed out during the round.**

- All work during this round must be done as an individual. **No conversation is allowed during this round.**

Written Test Problems

(1) What is the integer closest to $\sqrt[5]{2024}$?

(A) 4 (B) 5 (C) 44 (D) 45 (E) None of these

(2) The price of a t-shirt, prior to sales tax, is \$18.74. After applying sales tax of t%, where t is a whole number, the total price (rounded to the nearest cent, as usual) of the shirt is \$20.24. What is t?

(A) 5 (B) 6 (C) 7 (D) 8 (E) None of these

(3) What is the sum of all positive divisors (aka factors) of 2024?

(A) 4320 (B) 2592 (C) 2296 (D) 42 (E) None of these

(4) Kate has taken four tests in her calculus class so far this semester. Her average test score is 84, and her lowest test score is 66. The class has a test replacement policy: if your fifth test score beats your lowest test score, then your fifth score counts twice and your lowest score does not count at all. If Kate scores a 100 on her fifth test, what will be her adjusted average test score for the semester?

(A) 94 (B) 92 (C) 90 (D) 89 (E) None of these

(5) What is the minimum value of n such that the sum of the measures of the interior angles of a regular n-gon exceeds 2024°?

(A) 11 (B) 12 (C) 13 (D) 14 (E) None of these

(6) What is the minimum value of n such that the sum of the measures of the interior angles of a regular n-gon exceeds 2024 radians?

(A) 323 (B) 325 (C) 647 (D) 649 (E) None of these

(7) What is the mean of the first 20 positive perfect cubes?

(A) 21/2 (B) 287/2 (C) 9261/8 (D) 2205 (E) None of these

(8) What positive integer n satisfies $\log_{10}(25!) - \log_{10}(23!) = 2 + \log_{10}(n)$?

(A) 4 (B) 5 (C) 6 (D) 7 (E) None of these

(9) What is the sum of all integers n such that $\frac{20}{24-n}$ is an integer?

(A) 288 (B) 244 (C) 144 (D) 102 (E) None of these

(10) A bowl contains three identical-looking coins. Two of the coins are fair, so they land heads with probability 1/2, while the other coin is unfair and lands heads with probability 1/3. A coin is randomly selected from the bowl and flipped. What is the probability of heads?

(A) 8/17 (B) 4/9 (C) 5/12 (D) 1/3 (E) None of these

(11) A bowl contains three identical-looking coins. Two of the coins are fair, so they land heads with probability 1/2, while the other coin is unfair and lands heads with probability 1/3. A coin is randomly selected from the bowl and flipped twice. Given that both flips are tails, what is the probability that the selected coin is unfair?

(A) 8/17 (B) 4/9 (C) 5/12 (D) 1/3 (E) None of these

(12) What is the maximum value of $10^{2x^3-x^6+1}$?

(A) 1/10 (B) 1 (C) 10 (D) No maximum (E) None of these

(13) How many positive real solutions are there to the equation $\log_{20}(x) = \sin(\pi x)$?

(A) 1 (B) 19 (C) 20 (D) 21 (E) None of these

(14) Consider the equation

$$\sqrt[4]{x^3} + 2\sqrt{x} = 35\sqrt[4]{x}.$$

If n is the number of real solutions to the equation, and S is the sum of the real solutions to the equation, what is $n + S$?

(A) 626 (B) 627 (C) 3028 (D) 3029 (E) None of these

(15) How many integers between 1 and 2024, inclusive, have no common factors with 30 that are greater than 1?

(A) 536 (B) 540 (C) 636 (D) 640 (E) None of these

(16) How many real solutions are there to the equation

$$5\cos^4(x) + 2 = 11\cos^2(x)$$

with $0 \leq x < 2\pi$?

(A) 0 (B) 2 (C) 4 (D) 8 (E) None of these

(17) What is the product of all solutions to the equation

$$8\cos^4(x) + 3 = 10\cos^2(x)$$

with $0 \leq x < 2\pi$?

(A) $\frac{1}{324}\pi^4$ (B) $\frac{40}{81}\pi^4$ (C) $\frac{25}{26224}\pi^8$ (D) $\frac{13475}{110592}\pi^8$ (E) None of these

(18) The function $\arcsin(x)$ (also denoted $\sin^{-1}(x)$) is defined as the unique angle θ satisfying $-\pi/2 \leq \theta \leq \pi/2$ and $\sin(\theta) = x$. What is the domain of $\arcsin(8 - x^2)$?

(A) $[-1, 1]$ (B) $[7, 9]$ (C) $[-9, -7] \cup [7, 9]$ (D) $[\sqrt{7}, 3]$ (E) None of these

(19) In a room with 82 people, what is the expected value of the number of groups of three people all born in the same month? Assume people are born in each month with equal probability. (Clarification: each distinct group of three people counts separately, so, for example, a group of five people all born in January would count as ten distinct groups of three people all born in January.)

(A) 41/72 (B) 369/16 (C) 615 (D) 88560 (E) None of these

(20) A *k-term arithmetic progression* (kAP for short) with common difference d is a set of numbers the form

$$\{x, x + d, \ldots, x + (k - 1)d\}.$$

For example, $\{3, 10, 17, 24\}$ is a 4AP with common difference 7. What is the sum of the terms of the 6AP consisting entirely of prime numbers with the smallest possible largest element?

(A) 120 (B) 132 (C) 492 (D) 666 (E) None of these

Written Test Solutions

(1) Since $4^5 = 1024$ and $5^5 = 3125$, we know $\sqrt[5]{2024}$ is between 4 and 5. Since 2024 is slightly closer to 4^5 than 5^5, one might guess this implies the answer is 4, but this is faulty reasoning, as fifth roots get closer and closer together as numbers increase. One way to tell whether the answer is 4 or 5 is to determine whether $4.5^5 = 9^5/2^5$ is above or below 2024. Since $9^5 = 59049$ and $2^5 = 32$, it is clear that $4.5^5 < 2000$, so the answer is 5, which is **(B)**.

(2) The sales tax is $20.24 - 18.74 = 1.50$, and $1.50/18.74$ is very close to .08, so the answer is **(D)**.

(3) The prime factorization of 2024 is $2^3 \cdot 11 \cdot 23$, which incidentally was given in the first ciphering problem. One could do a brute force calculation by determining all 16 positive divisors of 2024, or utilize the fact that adding divisors is multiplicative, so the answer is $(1 + 2 + 4 + 8)(1 + 11)(1 + 23) = 4320$, which is **(A)**.

(4) Kate's first four test scores add to $4 \cdot 84 = 336$, and her lowest is 66, so the other three add to $336 - 66 = 270$. Adding a 100 counting double gives her five test scores adding to 470, for an average of $470/5 = 94$, which is **(A)**.

(5) The formula for the sum of the measures of the interior angles of a regular n-gon, in degrees, is $180(n - 2)$. Setting this equal to 2024 gives $n = 2 + (2024/180) \approx 13.2$, but since the number of sides must be a whole number, the answer is 14, so **(D)**.

(6) The formula for the sum of the measures of the interior angles of a regular n-gon, in radians, is $\pi(n - 2)$. Setting this equal to 2024 gives $n = 2 + (2024/\pi) \approx 646.3$, but since the number of sides must be a whole number, the answer is 647, so **(C)**.

(7) One could compute and add together all 20 perfect cubes, or use the shortcut that the sum of the first n cubes is the square of the sum of the first n positive integers. In this case, $1+2+\cdots+20 = 20(21)/2 = 210$, so $1+8+27+\cdots+20^3 = 210^2$, hence the mean is $210^2/20 = 2205$, which is **(D)**.

(8) Using rules of logarithms, the left hand side is $\log_{10}(25!/23!) = \log_{10}(600)$. Subtracting $\log_{10}(n)$ from both sides gives $\log_{10}(600/n) = 2$. Raising 10 to both sides gives $600/n = 100$, hence $n = 6$, which is **(C)**.

(9) The qualifying values of n are $4, 14, 19, 20, 22, 23, 25, 26, 28, 29, 34, 44$, which add to 288, so **(A)**.

(10) The probability of selecting a fair coin and flipping heads is $(2/3)(1/2) = 1/3$, and the probability of selecting the unfair coin and flipping heads is $(1/3)(1/3) = 1/9$, so the total probability of heads is $1/3 + 1/9 = 4/9$, which is **(B)**.

(11) The probability of selecting a fair coin and flipping two tails is $(2/3)(1/4) = 1/6$, and the probability of selecting the unfair coin and flipping two tails is $(1/3)(4/9) = 4/27$, so the total probability of flipping two tails is $1/6 + 4/27 = 17/54$, of which $8/54$ corresponds to the unfair coin. Therefore, the conditional probability we desire is $(8/54)/(17/54) = 8/17$, which is **(A)**.

(12) Completing the square, we can rewrite the exponent as $2 - (x^3 - 1)^2$, which is equal to 2 when $x = 1$, and is otherwise less than 2. Therefore, the maximum value of the given function is $10^2 = 100$, so the answer is **(E)**.

(13) The left hand side is strictly increasing, equals -1 when $x = 1/20$, and equals 1 when $x = 20$, so all intersections must occur in the interval $1/20 < x < 20$. The right hand side equals 1 whenever $x = 2k + 1/2$, and equals -1 whenever $x = 2k + 3/2$, where k is an integer, so it must intersect with the left side somewhere between $2k + 1/2$ and $2k + 3/2$, and then again between $2k + 3/2$ and $2(k + 1) + 1/2$, for $k = 0, \ldots, 8$, for a total of 18 intersections. One final intersection occurs between $x = 18.5$ and $x = 19.5$, but then the right side remains negative until settling at 0 when $x = 20$, for a total of 19 intersections, which is **(B)**.

(14) Subtracting the right side to the left, and factoring out $x^{1/4}$, gives

$$x^{1/4}(x^{1/2} + 2x^{1/4} - 35) = u(u^2 + 2u - 35) = u(u - 5)(u + 7) = 0,$$

where $u = x^{1/4}$. Therefore, $u = 0$, $u = 5$, or $u = -7$. The first two yield solutions $x = 0$ and $x = 5^4 = 625$, respectively, but the third does not yield a solution, as $x^{1/4}$ is nonnegative. Therefore, $n = 2$ and $S = 625$, and the final answer is 627, so **(B)**.

(15) Every block of 30 consecutive integers contains exactly 8 integers that are relatively prime (meaning no common factors greater than 1) to 30, the ones which when divided by 30 give remainders $1, 7, 11, 13, 17, 19, 23, 29$. Between 1 and 2010, inclusive, there are 67 blocks of 30 consecutive integers, so $67 \cdot 8 = 536$ of them are relatively prime to 30. Finally, the integers between 2011 and 2024, inclusive, include the remainders $1, 7, 11, 13$, so four of them are relatively prime to 30, for a total of 540, which is **(B)**.

(16) Letting $u = \cos^2(x)$ and subtracting the right side to the left gives

$$5u^2 - 11u + 2 = (5u - 1)(u - 2) = 0,$$

so $u = 1/5$ or $u = 2$. The latter is impossible, and the former gives $\cos(x) = \pm 1/\sqrt{5}$. Each of the two values yields two solutions, for a total of 4, which is **(C)**.

(17) Letting $u = \cos^2(x)$ and subtracting the right side to the left gives

$$8u^2 - 10u + 3 = (4u - 3)(2u - 1) = 0,$$

so $u = 3/4$ or $u = 1/2$. The former gives $\cos(x) = \pm\sqrt{3}/2$, which has solutions $x = \pi/6,\ 5\pi/6,\ 7\pi/6,\ 11\pi/6$, and the latter gives $\cos(x) = \pm 1/\sqrt{2}$, which has solutions $x = \pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4$. Multiplying all eight solutions together and simplifying gives $\frac{13475}{110592}\pi^8$, which is **(D)**.

(18) For there to exist θ with $\sin(\theta) = 8 - x^2$, we must have $-1 \leq 8 - x^2 \leq 1$, hence $7 \leq x^2 \leq 9$, which means either $\sqrt{7} \leq x \leq 3$ or $-3 \leq x \leq -\sqrt{7}$, for a final answer of $[-3, -\sqrt{7}] \cup [\sqrt{7}, 3]$, so the answer is **(E)**.

(19) Each group of three people share a common birth month with probability $(1/12)^2 = 1/144$, as the first person can be born in any month, and then the next two people have to match the first. There are $\binom{82}{3}$ distinct groups of three people, so by additivity of expected value the answer is $\binom{82}{3}/144 = (82)(81)(80)/(6 \cdot 144) = 615$, which is **(C)**.

(20) If the common difference of a 6AP fails to be divisible by 2, 3, and 5, then the 6AP will contains two multiples of 2, or two multiples of 3, or two multiples of 5, hence it will contain at least one composite number. Therefore, the smallest possible common difference of a 6AP of primes is $(2)(3)(5) = 30$. Further, starting at $x = 7$ and using $d = 30$ turns out to work, yielding $\{7, 37, 67, 97, 127, 157\}$, which are all prime. Since 30 is the smallest step size, and starting at anything smaller than 7 does not work, this is the unique 6AP that we are looking for, so the answer is $7 + 37 + 67 + 97 + 127 + 157 = 492$, which is **(C)**.