2025 Millsaps College
High School Mathematics Competition

Ciphering Round **Solutions**
10 Problems/3 Minutes Each

- All problems are free response, and 10 points are awarded for each correct answer
- The only things allowed out during the round are the pages from this packet, writing utensils, and scratch paper. In particular, **no calculators or electronic devices of any kind are allowed out during the round.**
- Problems will be worked, and then collected, one at a time. **Do not look at the next page in the packet until directed by a proctor to do so.**
- All work during this round must be done as an individual. **No conversation is allowed during this round.**

Ciphering Round Problems

(1) For real numbers a and b, we define the operation $a\#b = (a+b)^2$. What is $(2\#3)\#20$?

(2) On October 18, 2025, Monte Quinn of Curry College set an all-division NCAA football record with 522 rushing yards in a single game, which he accomplished in only 20 carries. If he carries the ball 25 times in his next game, how many yards per carry must he average in the next game to achieve 1000 total yards for the two games? Express your answer in decimal form.

(3) A pizzeria has two party packages. The first packge is 3 small pizzas (10-inch diameter) and 4 medium pizzas (15-inch diameter) for \$60. The second package is 4 large pizzas (20-inch diameter). If the packages are priced equivalently by area, what should the price of the second package be?

(4) What is $\tan(2025^\circ)$?

(5) If $(2x+7)^4 = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ for all real numbers x, what is

$$a_4 - a_3 + a_2 - a_1 + a_0 \ ?$$

(6) Until 1969, the United States Treasury issued bills in denominations \$1, \$2, \$5, \$10, \$20, \$50, \$100, \$500, \$1000, \$5000, and \$10000. How many total dollars would you have if, for each denomination \$$n$, you had n bills (in other words, one \$1 bill, two \$2 bills, five \$5 bills, . . . , five thousand \$5000 bills, and ten thousand \$10000 bills)?

(7) Roswell High School is playing Blessed Trinity High School in a women's basketball game. Roswell has 10 players on their roster, while Blessed Trinity has 8. Kate and Anna are sisters and play for Roswell, but their other sister, Lilly, plays for Blessed Trinity. Each team selects a starting lineup of five players at random. What is the probability that *all three* of the sisters start the game? Expressed your answer as a reduced fraction.

(8) What is the least positive integer that can be written as a sum of two integer cubes (negatives allowed) in two different ways (other than just swapping the order)?

(9) What is the sum of all solutions to

$$8\cos^2(x)\sin(x) + 3 = 4\cos^2(x) + 6\sin(x)$$

with $0 \le x \le \pi$? Provide an exact answer in terms of π.

(10) A class has $n \ge 2$ students, and every student high-fives every student (other than themselves) exactly once. The total number of high-fives is divisible by 2025. What is the least possible value of n?

Ciphering Round Solutions

(1) $(2\#3)\#20 = (2+3)^2\#20 = 25\#20 = (25+20)^2 = 45^2 = \mathbf{2025}$

(2) He must gain $1000-522 = 478$ yards in 25 carries, so he must average $478/25 = \mathbf{19.12}$ yards per carry.

(3) One could work directly with the formula for the area of a circle, but in fact the answer does not depend on the pizzas being circular, only that area is two-dimensional. Since the small has half the diameter of the large, it has $(1/2)^2 = 1/4$ the area. Since the medium has $3/4$ the diameter of the large, it has $(3/4)^2 = 9/16$ the area. So, three smalls and four mediums have the same area as $3(1/4) + 4(9/16) = 3$ large pizzas. Since the first package costs \$60, that is \$20 per large pizza area, hence the second package should cost $20 \cdot 4 =$**\$80**.

(4) Since $2025 = 5 \cdot 360 + 225$, we know $\tan(2025^\circ) = \tan(225^\circ)$, as the two angles differ by five complete circles. Further, $225 = 180 + 45$, so the angle 225° lies in the third quadrant of the unit circle, where both the horizontal coordinate (cosine) and the vertical coordinate (sine) are negative, on the diagonal. In particular, even if we did not know the precise values of $\cos(225^\circ)$ and $\sin(225^\circ)$, we know they are equal to each other (they happen to be $1/\sqrt{2}$). Therefore,

$$\tan(2025^\circ) = \tan(225^\circ) = \sin(225^\circ)/\cos(225^\circ) = \mathbf{1}.$$

(5) Letting $P(x) = (2x+7)^4$, the key observation is that $a_4 - a_3 + a_2 - a_1 + a_0 = P(-1)$, so the answer is $(2(-1)+7)^4 = 5^4 = \mathbf{625}$.

(6) $1^2 + 2^2 + 5^2 + 10^2 + 20^2 + 50^2 + 100^2 + 500^2 + 1000^2 + 5000^2 + 10000^2 =$ **\$126,263,030**

(7) The total number of Roswell starting lineups is $\binom{10}{5}$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient 'n choose k'. Of these, $\binom{8}{3}$ lineups include both Kate and Anna, since their three fellow starters must be chosen from their eight remaining teammates. Similarly, Blessed Trinity has $\binom{8}{5}$ lineups, of which $\binom{7}{4}$ include Lilly. Since the teams choose their lineups independent from each other, the probability that all three sisters starts is

$$p = \frac{\binom{8}{3}}{\binom{10}{5}} \cdot \frac{\binom{7}{4}}{\binom{8}{5}}.$$

Canceling common factors in each binomial coefficient formula gives $\binom{8}{3} = \binom{8}{5} = 8 \cdot 7$ (in fact $\binom{n}{k}$ always equals $\binom{n}{n-k}$), $\binom{7}{4} = 7 \cdot 5$, and $\binom{10}{5} = 9 \cdot 7 \cdot 4$, so $\binom{8}{3}$ and $\binom{8}{5}$ cancel and we have

$$p = \frac{7 \cdot 5}{9 \cdot 7 \cdot 4} = \frac{\mathbf{5}}{\mathbf{36}}.$$

(8) $\mathbf{91} = 3^3 + 4^3 = 6^3 + (-5)^3$; you can confirm this is the smallest by listing out the first six cubes, looking at sums and differences, and noting that

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

exceeds 91 if $x > y > 0$ are integers and $x > 6$.

(9) Moving everything to one side and factoring we get

$$8\cos^2 \sin(x) - 4\cos^2(x) - 6\sin(x) + 3 = (4\cos^2(x) - 3)(2\sin(x) - 1) = 0,$$

so $\cos(x) = \pm\sqrt{3}/2$ or $\sin(x) = 1/2$. For $0 \le x \le \pi$, these two conditions coincide, each occurring at $x = \pi/6$ and $x = 5\pi/6$, so the answer is $\pi/6 + 5\pi/6 = \pi$.

(10) The total number of high-fives is $\binom{n}{2} = n(n-1)/2$, which is divisible by $2025 = 25 \cdot 81$ if and only if $n(n-1)$ is. Since n and $n-1$ are relatively prime, it must be the case that one of n or $n-1$ is divisible by 25, and the other is divisible by 81. In other words, either n is a multiple of 25 that is congruent to 1 modulo 81, or n is a multiple of 81 that is 1 modulo 25. The first time the former occurs is $n = 13 \cdot 25 = 325$, while the first time the latter occurs is $n = 21 \cdot 81 = 1701$, so the answer is **325**.

For those interested, the easiest way to find that $21 \cdot 81$ is the first for the latter is to consider the multiplicative inverse of $81 \equiv 6 \pmod{25}$. For the former, I just did a brute force list.