2025 Millsaps College
High School Mathematics Competition

Team Round **Solutions**
5 Problems/60 Minutes

- All problems are free response, and 40 points are awarded for each correct answer.

- The only things allowed out during the round are the pages from this packet, writing utensils, and scratch paper. In particular, **no calculators or electronic devices of any kind are allowed out during the round.**

- This round is collaborative. Teammates are encouraged to work together and communicate during the round, but there should not be any communication between teams.

Team Round Problems

(1) A five-card hand is drawn at random from a standard 52-card deck (13 of each suit: clubs, diamonds, hearts, spades). What is the probability that the hand contains at least four spades? Express your answer as a reduced fraction.

(2) Three *Jeopardy!* contestants enter the *Final Jeopardy* clue with scores as follows: Tom has $22800, Kristina has $10800, and Shaan has $15600. On the final clue, each player can wager any whole number of dollars from $0 up to all of their money. They gain their wager if they respond correctly, but lose their wager if they respond incorrectly.

Whoever has the most money after *Final Jeopardy* wins the game and the money they have accrued during the game. The second-place contestant wins $3000, while the third-place contestant wins $2000, regardless of their actual scores. (The second and third place prizes do not come into play in this problem.)

Tom wants to 'control his destiny', meaning he wants to guarantee that if he responds correctly, he will win (not tie) the game. Subject to that constraint, he wagers as little as possible. Call this *strategy A*.

Shaan assumes that Tom will employ the strategy A, as it is the most common one for players in first place entering *Final Jeopardy*. Shaan knows that if Tom employs strategy A and responds correctly, then Shaan cannot win, so Shaan decides to focus on the scenario that Tom responds incorrectly. Shaan figures that if Tom responds incorrectly, the clue is likely difficult, so he wants to ensure that if *both* he and Tom respond incorrectly, then Shaan will beat (not tie) Tom. Subject to this constraint, he wagers as much as possible, in order to improve his chance of beating Kristina with a correct response. Call this *Strategy B*.

Kristina assumes that Tom will employ Strategy A and Shaan will employ Strategy B. She knows she cannot beat Tom if Tom responds correctly, so she focuses on the worst-case scenario in which she could still possibly win: Tom responds incorrectly, Shaan responds correctly. She wagers to ensure that if she responds correctly in this scenario, she will win (not tie) the game. Subject to this constraint, she wagers as little as possible, just in case her opponents' wagers surprise her.

What is the sum of the three constestants *Final Jeopardy* wagers?

(3) A *Gaussian integer* is a complex number of the form $z = a + bi$, where a and b are integers, and $i^2 = -1$. Just like any complex numbers, Gaussian integers $z = a + bi$ and $w = c + di$ can be multiplied as follows:

$$zw = (a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i.$$

A Gaussian integer z (other than ± 1 or $\pm i$) is called a *Gaussian prime* if whenever $z = uv$, with u and v Gaussian integers, either u or v must be ± 1 or $\pm i$.

Of the first four usual prime numbers $2, 3, 5, 7$, which one(s) are Gaussian primes?

(There could be anywhere from zero to four answers. Write your answers in a list separated by commas. If you think there are none, write 'NONE'.)

(4) For the purposes of this problem, a *tournament* is a round-robin competition between a collection of teams in which every team plays every team (other than themselves) exactly once, and every game has one winner and one loser (no ties). For example, the following describes a tournament with four teams, A, B, C, D:

$$A \text{ beats } B, \quad B \text{ beats } C, \quad C \text{ beats } A, \quad \text{all of } A, B, C \text{ beat } D.$$

A *transitive subtournament* of a tournament is a subset of the teams, in a particular order, such that team x beats team y if and only if x comes before y in the order. For example (A, B, D) and (C, A, D) are both transitive subtournaments of size 3 of the tournament above, but there is no transitive ordering of the full set of 4 teams.

Suppose a tournament has n teams, one of which is a 'superteam' that wins all of its games. What is the minimum value of n such that the tournament is *guaranteed* to have a transitive subtournament of size 4?

(5) Jacques the frog sits on the xy-plane at the origin $(0, 0)$. Each hop Jacques takes must cover a distance of exactly 65 units, and must land on a point (x, y) where x and y are both integers. What is the minimum number of hops Jacques can take in order to arrive at the point $(1410, 1410)$?

Team Round Solutions

(1) The total number of (unordered) hands is $\binom{52}{5}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient 'n choose k'. The number of hands with exactly four spades is $\binom{13}{4} \cdot 39$, as we must choose four spades and one non-spade, and the number of hands with five spades is $\binom{13}{5}$. So, the probability we want is

$$p = \frac{\binom{13}{4} \cdot 39 + \binom{13}{5}}{\binom{52}{5}}.$$

However, we still need to write this as a reduced fraction. Expanding the binomial coefficients, multiplying top and bottom by 5!, factoring out common factors from the numerator, and writing $5 \cdot 13 + 3 = 4 \cdot 17$, we have

$$p = \frac{\left(\frac{13 \cdot 12 \cdot 11 \cdot 10}{4!}\right)(39) + \left(\frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5!}\right)}{\left(\frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5!}\right)} = \frac{5 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 39 + 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}$$

$$= \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 3 \cdot (5 \cdot 13 + 3)}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 3 \cdot 4 \cdot 17}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}.$$

At this point, we can carefully cancel out all factors from the numerator except 11, and the remaining reduced fraction is **11/980**.

(2) Tom calculates that if Shaan wagers all of his \$15600 and responds correctly, Shaan will finish with \$31200. So, according to the specifications in the problem, Tom wagers enough to reach \$31201 with a correct response, which is \$(31201 − 22800) = \$8401.

Shaan assumes Tom wagers \$8401, so if Tom responds incorrectly, Tom will finish with \$(22800−8401) = \$14399. According to the specifications of the problem, Shaan chooses a wager such that he would drop to \$14400 with an incorrect response, which is \$(15600 − 14400) = \$1200.

Kristina assumes Tom wagers \$8401 and Shaan wagers \$1200. If Tom responds incorrectly and Shaan responds correctly, Shaan will be the leader of those two with \$(15600 + 1200) = \$16800. According to the specifications of the problem, Kristina wagers enough to reach \$16801 with a correct response, which is \$(16801 − 10800) = \$6001. Therefore, the final answer is \$(8401 + 1200 + 6001) = **\$15602**.

These were the actual contestants and scores entering Final Jeopardy on Friday, October 24. On the real show, Tom indeed wagered \$8401, while the others deviated. Only Shaan responded correctly, and he won the game. Interestingly, the fact that the answer is just \$2 more than Shaan's original score is a complete coincidence, due to the fact that the scores happen to satisfy the equation $8 \cdot \text{Shaan} = 5 \cdot \text{Tom} + \text{Kristina}$.

(3) A helpful observation is that $(a + bi)(a - bi) = a^2 + b^2$. This quickly tells us two of the usual primes that are *not* Gaussian primes, namely $2 = (1 + i)(1 - i)$ and $5 = (2 + i)(2 - i)$. (For those interested, this can be generalized with Fermat's two squares theorem to say that if a prime is 2 or congruent to 1 modulo 4, then it is *not* a Gaussian prime.)

Showing that the other two, 3 and 7, *are* Gaussian primes, is more subtle. A general argument is to define the *norm* of a Gaussian integer $z = a + bi$ by $N(z) = a^2 + b^2$, and noting that $N(uv) = N(u)N(v)$ for Gaussian integers u and v. So, if $3 = uv$ for Gaussian integers u and v, then $9 = N(3) = N(uv) = N(u)N(v)$, and if neither u nor v is ± 1 or $\pm i$, the only option is $N(u) = N(v) = 3$. However, 3 cannot be written as the sum of two integer squares, so it cannot be the norm of a Gaussian integer, so such a factorization of 3 is impossible. The same argument works for 7, and in fact any usual prime number p that is congruent to 3 modulo 4. So the answer to the problem is **3, 7**.

This generalizable approach is not necessary, you can play around with Gaussian integer multiplication in a more ad hoc way and convince yourself that 3 and 7 cannot be factored.

(4) First thing to note: to get a transitive subtournament of size 4, we just need a transitive subtournament of size 3 that does *not* include the superteam, because when we add the superteam to the front the subtournament will stay transitive. In particular, the answer to the problem is 1 more than the answer to a simpler problem: how many teams are needed to guarantee a transitive subtournament of size 3?

Note that three teams is not enough, because if they form a cycle (e.g. A beats B, B beats C, C beats A), they cannot be ordered transitively. However, with a fourth team D, since D plays three games, D must either lose at least two or win at least two (this is the *pigeonhole principle*). If two teams beat D (say A and B), then D is 'last place' in a transitive subtournament (A, B, D). If D beats two teams (say A and B), then D is 'first place' in a transitive subtournament (D, A, B). In any case, with four teams, there exists a three-team transitive subtournament. Therefore, the answer to the original problem is **5**.

For those interested, if we drop the assumption that there is a superteam, we are not guaranteed a four-team transitive subtournament until the number of teams reaches 8. A counterexample with $n = 7$ (due to Erdős and Moser in 1964) is to number the teams $0, 1, 2, 3, 4, 5, 6$, and have each team x beat teams $x + 1, x + 2, x + 4 \pmod 7$.

(5) The first (and perhaps hardest) step is to determine which jumps are available to Jacques, or equivalently the collection of pairs (a, b) with a and b integers such that $a^2 + b^2 = 65^2$. The obvious ones are $(\pm 65, 0)$ and $(0, \pm 65)$, corresponding to Jacques jumping horizontally or vertically. Further, one can scale the known right triangles $(3, 4, 5)$ and $(5, 12, 13)$ to $(39, 52, 65)$ and $(25, 60, 65)$, respectively, giving Jacques the jumps $(\pm 39, \pm 52), (\pm 52, \pm 39)$ and $(\pm 25, \pm 60), (\pm 60, \pm 25)$. There are more, though, because 65 is also the hypotenuse of two primitive (meaning not a scaling of a smaller one) right triangles with integer side lengths (called *Pythagorean triples*). The quickest way to find them is using the parametrization of primitive pythagorean triples (a, b, c) given by $a = u^2 - v^2$, $b = 2uv$, $c = u^2 + v^2$, where $u > v$, $u + v$ is odd, and u and v are relatively prime. Since $65 = 8^2 + 1^2 = 7^2 + 4^2$, we get the additional pairs $(a, b) = (\pm 63, \pm 16), (\pm 16, \pm 63)$ and $(a, b) = (\pm 33, \pm 56), (\pm 56, \pm 33)$.

By doing two 'reflected jumps' in succession (for example $(39, 52)$ then $(52, 39)$), Jacques can efficiently travel (x, x) for $x = 65, 79, 85, 89, 91$. Since the goal is to reach $(1410, 1410)$, a potential strategy is to write 1410 as a sum of numbers in that list, using as few as possible. Unfortunately 1410 is not divisible by 91, but $91 \cdot 12 = 1092$, and the remainder $1410 - 1092 = 318 = 89 + 85 + 79 + 65$. This gives a path for Jacques with a total of $24 + 2 + 2 + 2 + 2 = 32$ jumps, as follows: twelve each of $(39, 52)$ and $(52, 39)$, then one each of $(33, 56), (56, 33), (25, 60), (60, 25), (16, 63), (63, 16), (0, 65), (65, 0)$.

But how do we know this is optimal? At the end of Jacque's trip, the sum of his x and y coordinates is $1410 + 1410 = 2820$. However, each jump contributes at most $52 + 39 = 91$ to the sum (all others contribute less), so the number of jumps is at least $\lceil 2820/91 \rceil = \lceil 30.99 \rceil = 31$. To see that 31 is impossible, note that if even one of the first 31 jumps contributes less than 91, we have $x + y \leq 91 \cdot 30 + 89 = 2819 < 2820$, so the only way $x + y$ is big enough is if they are *all* $(52, 39)$ or $(39, 52)$, but in that case we would have $x + y = 91 \cdot 31 = 2821 > 2820$, so that does not work either.

We have now shown that 32 jumps is possible for Jacques, while anything less than 32 is impossible, so the minimum is exactly **32** jumps.

Credit Note: This is a modification of Problem A1 of the 82nd William Lowell Putnam Competition, December 2021.